PROCEEDINGS

OF THE

AMERICAN SOCIETY OF CIVIL ENGINEERS

VOL. 68 APRIL, 1942 No. 4, PART 1

TECHNICAL PAPERS

AND

DISCUSSIONS

Published monthly, except July and August, at Prince and Lemon Streets, Lancaster, Pa., by the American Society of Civil Engineers. Editorial and General Offices at 33 West Thirty-ninth Street, New York, N. Y. Reprints from this publication may be made on condition that the full title of Paper, name of Author, page reference, and date of publication by the Society, are given.

Entered as Second-Class Matter, September 23, 1937, at the Post Office at Lancaster, Pa., under the Act of March 3, 1879. Acceptance for mailing at special rate of postage provided for in Section 1103, Act of October 3, 1917, authorized on July 5, 1918.

Subscription (if entered before January 1) $8.00 per annum. Price $1.00 per copy

CURRENT PAPERS AND DISCUSSIONS

		Discussion closes
Consumptive Use of Water for Agriculture. *Robert L. Lowry, Jr., and Arthur F. Johnson*	Apr., 1941	
Discussion	June, Sept., Oct., 1941	Closed*
Operation Experiences, Tygart Reservoir. *Robert M. Morris and Thomas L. Reilly*	Apr., 1941	
Discussion	Oct., 1941, Jan., 1942	Closed*
The Surveyor and the Law. *A. H. Holt*	May, 1941	
Discussion	Oct., Dec., 1941	Closed*
Extensometer Stress Measurements, North Avenue Bridge, Chicago, Ill. *Lawrence T. Smith and Paul Lillard*	May, 1941	
Discussion	June, Oct., 1941	Closed*
Traffic Engineering as Applied to Rural Highways. *Milton Harris*	May, 1941	
Discussion	Sept., Oct., Dec., 1941, Jan., 1942	Closed*
Pile-Driving Formulas: Progress Report of the Committee on the Bearing Value of Pile Foundations	May, 1941	
Discussion	Sept., Oct., Nov., Dec., 1941, Jan., Feb., Mar., 1942	Closed*
Salts in Irrigation Water. *Raymond A. Hill*	June, 1941	
Discussion	Sept., Oct., Dec., 1941	Closed*
Cost of Public Services in Residential Areas. *F. Dodd McHugh*	June, 1941	
Discussion	Oct., Nov., 1941, Feb., 1942	Closed*
Compaction of Cohesionless Foundation Soils by Explosives. *A. K. B. Lyman*	May, 1941	
Discussion	Oct., 1941, Jan., Feb., 1942	Closed*
A Direct Method of Flood Routing. *C. O. Wisler and E. F. Brater*	June, 1941	
Discussion	Oct., Nov., 1941	Closed*
An Investigation of Plate Girder Web Splices. *J. M. Garrelts and I. E. Madsen*	June, 1941	
Discussion	Sept., Oct., 1941	Closed*
Design and Construction of San Gabriel Dam No. 1. *Paul Baumann*	Sept., 1941	
Discussion	Nov., Dec., 1941, Jan., Feb., Mar., Apr., 1942	Closed*
Pipe-Line Flow of Solids in Suspension: A Symposium	Oct., 1941	
Discussion	Oct., Dec., 1941, Mar., Apr., 1942	Closed*
Fundamental Aspects of the Depreciation Problem: A Symposium	Nov., 1941	
Discussion	Nov., Dec., 1941, Feb., Mar., Apr., 1942	May, 1942
Stability of Granular Materials. *R. E. Glover and F. E. Cornwell*	Nov., 1941	
Discussion	Nov., 1941, Mar., Apr., 1942	May, 1942
Timber Friction Pile Foundations. *Frank M. Masters*	Nov., 1941	
Discussion	Feb., Apr., 1942	May, 1942
Bending Moments in the Walls of Rectangular Tanks. *Dana Young*	Nov., 1941	
Discussion	Feb., 1942	May, 1942
Energy Loss at the Base of a Free Overfall. *Walter L. Moore*	Nov., 1941	
Discussion	Jan., Apr., 1942	May, 1942
Design of St. Georges Tied Arch Span. *J. M. Garrelts*	Dec., 1941	
Discussion	Feb., Apr., 1942	May, 1942
Allocation of the Tennessee Valley Authority Projects. *Theodore B. Parker*	Dec., 1941	
Discussion	Feb., Mar., 1942	May, 1942
Development of Transportation in the United States. *J. E. Teal*	Dec., 1941	
Discussion	Feb., Mar., Apr., 1942	May, 1942
Lateral Stability of Unsymmetrical I-Beams and Trusses in Bending. *George Winter*	Dec., 1941	
Discussion	Feb., Mar., 1942	May, 1942
Drainage of Leveed Areas in Mountainous Valleys. *Gordon R. Williams*	Jan., 1942	
Discussion	Jan., Apr., 1942	May, 1942
Hydraulic Design of Drop Structures for Gully Control. *B. T. Morris and D. C. Johnson*	Jan., 1942	
Discussion	Apr., 1942	May, 1942
Analytical and Experimental Methods in Engineering Seismology. *M. A. Biot*	Jan., 1942	
Discussion	Apr., 1942	May, 1942
Effect of Variation of Elastic Characteristics on Static Unknowns. *A. Hrennikoff*	Jan., 1942	
Discussion	Apr., 1942	May, 1942
Design of Sign Letter Sizes. *Adolphus Mitchell and T. W. Forbes*	Jan., 1942	May, 1942
Protective and Remedial Measures for Sanitary and Public Health Engineering Services: Progress Report of the Sanitary and Public Health Engineering Division of the National Committee of the Society, on Civilian Protection in War Time	Jan., 1942	
Discussion	Mar., Apr., 1942	May, 1942
The Grease Problem in Sewage Treatment. *Almon L. Fales and Samuel A. Greeley*	Feb., 1942	
Discussion	Feb., 1942	June, 1942
Unusual Events and Their Relation to Federal Water Policies. *W. G. Hoyt*	Feb., 1942	
Discussion	Apr., 1942	June, 1942
Reduction of Mineral Content in Water with Organic Zeolites. *R. F. Goudey*	Feb., 1942	
Discussion	Feb., Apr., 1942	June, 1942
Resistance of Soft Fill to Static Wheel Loads. *W. Watters Pagon*	Feb., 1942	June, 1942
Technique of Determining Shearing Strength of Soils: Progress Report of a Special Committee of the Soil Mechanics and Foundations Division on the Technique of Soil Tests	Feb., 1942	
Discussion	Feb., Apr., 1942	June, 1942
Viscosity and Surface Tension Effects on V-Notch Weir Coefficients. *Arno T. Lenz*	Mar., 1942	July, 1942
Hydrodynamics of Model Storm Sewer Inlets Applied to Design. *G. S. Tapley*	Mar., 1942	July, 1942
Basic Economic Considerations Affecting Single and Multiple Purpose Projects. *Otis M. Page*	Mar., 1942	July, 1942
Early Contributions to Mississippi River Hydrology. *C. S. Jarvis*	Mar., 1942	July, 1942
Model Tests on Structures for Hydroelectric Developments. *L. M. Davis*	Apr., 1942	
Discussion	Apr., 1942	Aug., 1942

NOTE.—The closing dates herein published are final except when names of prospective discussers are registered for special extension of time.

* Publication of closing discussion pending.

CONTENTS FOR APRIL, 1942

PAPERS

REPORTS

DISCUSSIONS

CONTENTS FOR APRIL, 1942 (*Continued*)

For Index to all Papers, the discussion of which is current in PROCEEDINGS, *see page 2*

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

PAPERS

STRESS CONCENTRATIONS IN PLATES LOADED OVER SMALL AREAS

By H. M. Westergaard,[1] M. Am. Soc. C. E.

Synopsis

If a considerable pressure is applied within a small area as a transverse load on a plate or slab, a concentration of stress will occur within and around that area. This stress concentration differs in type from that observed at fillets, re-entrant corners, or holes in solids; it is caused by concentration of the load. If the load on the plate or slab is not too close to any lines or points of concentrated support or to any edge and is neither too concentrated nor too dispersed, and if the material of the plate or slab is homogeneous, isotropic, and elastic, the important features of the stress concentration produced by the load can be expressed conveniently and with good approximation by means of six coefficients, all of which are pure numbers. Three of these—B, C, and D—are called "place coefficients"; they depend on the place of the load, but not on its manner of distribution over the small loaded area. The remaining three—K, S, and T—are called "area coefficients"; they depend on the size and shape of the loaded area and on the manner of distribution of the load over that area, but are independent of the place of the load. When the place coefficients have been determined in m cases and the area coefficients in n cases, solutions are thereby made available in m times n combined cases.

Six examples demonstrate how the place coefficients may be obtained from solutions already available for loads concentrated at a point. Area coefficients are derived in twelve cases.

When the load is concentrated within a very small area, special corrections are needed. The most important of these can be made by replacing the place coefficient K by a "substitute coefficient" K'.

A final, numerical example shows the use of place coefficients and area coefficients in determining stresses in a concrete pavement on elastic subgrade

Note.—Written comments are invited for immediate publication; to insure publication the last discussion should be submitted by August, 1942.

[1] Dean, Graduate School of Eng., and Gordon McKay Prof. of Civ. Eng., Harvard Univ., Cambridge, Mass.

under a wheel load which is distributed through dual tires over the combined area of two ellipses.

Notation

For convenience the plate or slab is assumed horizontal, with the load acting as a downward pressure on the top. The following notation is used:

P = total load;
p = load per unit of area;
A = loaded area;
h = thickness of the plate or slab;
μ = Poisson's ratio;
N = measure of stiffness of the plate or slab, defined by Eq. 2;
x, y = horizontal rectangular coordinates;
r, θ = corresponding polar coordinates;
u, v = particular values of x, y;
R, α = relative polar coordinates, defined by Eqs. 7, and corresponding to the relative rectangular coordinates $x - u$ and $y - v$;
ζ = deflection at point x, y, positive downward;
m_x, m_y, m_{xy} = bending moments and twisting moment at point x, y in the directions of x and y, per unit of width of section, positive when the corresponding stresses at the bottom of the plate or slab are positive; measurable in inch-pounds per inch, or in pounds;
$\sigma_x, \sigma_y, \tau_{xy}$ = horizontal normal stresses and shearing stress at the bottom of the plate or slab in the directions of x and y;
c = distance chosen arbitrarily;
B, C, D = place coefficients, defined through Eqs. 1, 4, 8, and 9, and used by means of Eqs. 13 to 20; pure numbers;
K, S, T = area coefficients, defined by Eqs. 11 and 12, and used by means of Eqs. 13 to 20; pure numbers;
Δ = Laplace's operator, Eq. 3.

Logarithms are natural logarithms unless otherwise stated.

The Three Place Coefficients

The deflection ζ of a plate or slab at the point x,y due to a load P concentrated at the point u,v at the distance R from the point x,y may be written as a sum of terms, as follows: First, a term containing $R^2 \log R$, which is needed to account for the singularity at the point of the load;[2] second, a polynomial of second degree in x, y, u, and v; and third, a supplementing series of terms of higher than second degree in x, y, u, and v. When x, y, u, and v are kept small, the first term, containing $\log R$, will be a dominating one in the intended applications; the terms of second degree will be needed; but the supplementing series of terms of higher degree may be ignored. The number of different

[2] See, for example, "Die elastischen Platten," by A. Nádai (Julius Springer, Berlin), 1925, p. 202.

coefficients in the polynomial of second degree is restricted by Maxwell's law of reciprocal deflections, according to which the deflection at x,y due to P at u,v is equal to the deflection at u,v due to P at x,y. Thus one arrives at the following formula which still requires verification:

$$\zeta = \frac{P}{8\,\pi\,N}\Bigg[R^2\left(\log\frac{R}{c} - 1 - B\right) + c_0 + c_x\,(u+x) + c_y\,(v+y) + c_u\,u\,x + c_v\,v\,y + c'\,(u\,y + v\,x) - \frac{C}{2}\,(u^2 - v^2 + x^2 - y^2) - D\,(u\,v + x\,y)\Bigg] \qquad (1)$$

The constant N in Eq. 1 is the "measure of stiffness" of the plate or slab in bending; this constant is defined by the formula

$$N = \frac{E\,h^3}{12\,(1-\mu^2)} \qquad (2)$$

in which E is the modulus of elasticity, μ is Poisson's ratio, and h is the thickness of the plate or slab.

By applying Laplace's operator

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{\partial}{\partial R} + \frac{1}{R^2}\frac{\partial^2}{\partial \alpha^2} \qquad (3)$$

to Eq. 1 one finds

$$-\,N\,\Delta\,\zeta = \frac{P}{2\,\pi}\left(B - \log\frac{R}{c}\right) \qquad (4)$$

Transverse shears in plates or slabs per unit of length of section are first derivatives[3] of $-\,N\,\Delta\,\zeta$. By differentiating Eq. 4 with respect to R one finds the shear on a circular section with constant R. The value found is $-\,P/(2\,\pi\,R)$, which shows that the terms included in Eq. 1 account properly for the concentrated load P and for no other load within the region under consideration. This completes the verification of the form of Eq. 1.

The bending moments and the twisting moment in a plate or slab can be expressed by the formulas

$$\left.\begin{matrix} m_x \\ m_y \end{matrix}\right\} = -\,\frac{1+\mu}{2}\,N\,\Delta\,\zeta \mp \frac{1-\mu}{2}\,N\left(\frac{\partial^2\zeta}{\partial x^2} - \frac{\partial^2\zeta}{\partial y^2}\right) \qquad (5)$$

and

$$m_{xy} = -\,(1-\mu)\,N\,\frac{\partial^2\zeta}{\partial x\,\partial y} \qquad (6)$$

By taking ζ from Eq. 1 and noting that

$$x - u = R\cos\alpha \quad \text{and} \quad y - v = R\sin\alpha \qquad (7)$$

one finds

$$\left.\begin{matrix} m_x \\ m_y \end{matrix}\right\} = \frac{P}{4\,\pi}\left[(1+\mu)\left(B - \log\frac{R}{c}\right) \pm \frac{1-\mu}{2}\,(C - \cos 2\,\alpha)\right] \qquad (8)$$

[3] "Die elastischen Platten," by A. Nádai (Julius Springer, Berlin), 1925, p. 21.

and

$$m_{xy} = \frac{(1 - \mu) P}{8 \pi} (D - \sin 2 \alpha) \quad (9)$$

The three coefficients B, C, and D, which occur in Eqs. 1, 4, 8, and 9, will be considered as constants characteristic of the small region that is under consideration. The distance c can be chosen arbitrarily, but the choice of c will affect the value of B; the coefficient B attains its definite meaning after c has been chosen.

In the subsequent applications of Eqs. 8 and 9 it will not be necessary, unless it is specifically stated, to assume that x, y, u, and v are small; it is sufficient that the points x,y and u,v be contained within a small area; the distance R must remain small.

The Three Area Coefficients

Assume now that the load P is distributed over a small area A, so that the pressure is p per unit of area at u,v and

$$P = \int_A p \, dA \quad (10)$$

Then Eqs. 8 and 9 may be used with P replaced by $p \, dA$ and with integrations carried out over the area A.

It can now be seen that it will be expedient to introduce the following three "area coefficients," which are functions of x and y obtainable by integrations with respect to u and v over the loaded area:

$$K = - \frac{1}{P} \int_A p \, dA \log \frac{R}{c} \quad (11)$$

$$S = - \frac{1}{P} \int_A p \, dA \cos 2 \alpha \quad \text{and} \quad T = - \frac{1}{P} \int_A p \, dA \sin 2 \alpha \quad (12)$$

The coefficients K, S, and T are pure numbers. In terms of K, S, and T the integrals obtained from Eqs. 8 and 9 are expressed in the following formulas for the moments at point x,y due to the load distributed over the small area:

$$\left. \begin{matrix} m_x \\ m_y \end{matrix} \right\} = \frac{P}{4 \pi} \left[(1 + \mu) (B + K) \pm \frac{1 - \mu}{2} (C + S) \right] \quad (13)$$

and

$$m_{xy} = \frac{(1 - \mu) P}{8 \pi} (D + T) \quad (14)$$

The stresses at the bottom of the plate or slab corresponding to the moments in Eqs. 13 and 14 are

$$\left. \begin{matrix} \sigma_x \\ \sigma_y \end{matrix} \right\} = \frac{3 P}{2 \pi h^2} \left[(1 + \mu) (B + K) \pm \frac{1 - \mu}{2} (C + S) \right] \quad (15)$$

and

$$\tau_{xy} = \frac{3 (1 - \mu) P}{4 \pi h^2} (D + T) \quad (16)$$

The curvatures and twist corresponding to the moments in Eqs. 13 and 14 are

$$\left.\begin{array}{r} -\dfrac{\partial^2 \zeta}{\partial x^2} \\ -\dfrac{\partial^2 \zeta}{\partial y^2} \end{array}\right\} = \frac{P}{4\,\pi\,N}\left[B + K \pm \frac{C + S}{2}\right] \quad \text{(17)}$$

and

$$-\frac{\partial^2 \zeta}{\partial x\,\partial y} = \frac{P}{8\,\pi\,N}(D + T) \quad \text{(18)}$$

Finally, the strains and detrusion at the bottom of the plate or slab corresponding to the stresses in Eqs. 15 and 16 are

$$\left.\begin{array}{r} \epsilon_x \\ \epsilon_y \end{array}\right\} = \frac{P\,h}{8\,\pi\,N}\left[B + K \pm \frac{C + S}{2}\right] \quad \text{(19)}$$

$$\gamma_{xy} = \frac{P\,h}{8\,\pi\,N}(D + T) \quad \text{(20)}$$

When the load P is distributed uniformly over the area A, the area coefficients in Eqs. 11 and 12 can be stated in the slightly simpler form

$$K = -\frac{1}{A}\int_A dA \log\frac{R}{c} \quad \text{(21)}$$

$$S = -\frac{1}{A}\int_A dA \cos 2\,\alpha \quad \text{and} \quad T = -\frac{1}{A}\int_A dA \sin 2\,\alpha \quad \text{(22)}$$

The coefficient K defined by Eq. 11 or 21 is a logarithmic potential.[4] If p is constant within the loaded area, K may be produced as the gravitational potential created by a long uniform rod with the cross section shaped as the loaded area.

Differential Equations for the Area Coefficients

The deflections of a plate or slab are governed by Lagrange's equation[3]

$$\Delta^2 \zeta = p/N \quad \text{(23)}$$

When $\Delta\zeta$ is taken from Eq. 17, Eq. 23 becomes a differential equation for K; namely,

$$\Delta K = -\frac{2\,\pi\,p}{P} \quad \text{(24)}$$

By comparing the values of the third derivatives that may be obtained from Eqs. 17 and 18 one finds two more equations for K, S, and T; they are equations of compatibility, expressing requirements of consistent deformations. The equations are

$$\frac{\partial}{\partial x}(2\,K - S) = \frac{\partial T}{\partial y}, \qquad \frac{\partial}{\partial y}(2\,K + S) = \frac{\partial T}{\partial x} \quad \text{(25)}$$

[4] See, for example, "Foundations of Potential Theory," by Oliver Dimon Kellogg (Julius Springer, Berlin), 1929, pp. 62, 338, and 377.

The differential Eqs. 24 and 25 have two uses: One is the checking of results found by direct use of the definitions in Eqs. 11 and 12, or 21 and 22; the other use—which is actually more important—is in deriving formulas for K, S, and T. Examples of such derivations will be shown in a later section.

In the special group of cases in which the load is symmetrical around the origin of the coordinates, so that $p = p(r)$, K will be a function of r only, and S and T may be written in terms of a function $f(r)$, of r only, in the form

$$S = f(r) \cos 2\,\theta \qquad \text{and} \qquad T = f(r) \sin 2\,\theta \ldots\ldots\ldots\ldots (26)$$

Then Eq. 24 may be written in the special form

$$\frac{d^2K}{dr^2} + \frac{1}{r}\frac{dK}{dr} = -\frac{2\,\pi\,p}{P} \ldots\ldots\ldots\ldots (27)$$

and it is found that the two Eqs. 25 can be replaced by the single equation

$$\frac{df}{dr} + \frac{2\,f}{r} = 2\,\frac{dK}{dr} \ldots\ldots\ldots\ldots (28)$$

Note on the Special Significance of the Center of the Load

If the terms of third degree had been included in Eq. 1, linear terms in x, y, u, and v would have to be added in the expressions for the second derivatives of the deflection. Accordingly the moments in Eqs. 8 and 9 at x,y due to a load P at u,v would be supplemented by terms that can be written in the following form, the constants $c_1, c_2, \cdots c_{10}$ being of the dimension distance to the power -1:

$$\left.\begin{matrix} m'_x \\ m'_y \end{matrix}\right\} = P\,[(1+\mu)\,(c_1\,x + c_2\,y) \pm (1-\mu)\,(c_3\,x + c_4\,y)$$
$$+ (1+\mu)\,(c_5\,u + c_6\,v) \pm (1-\mu)\,(c_7\,u + c_8\,v)] \ldots\ldots\ldots (29)$$

and

$$m'_{xy} = (1-\mu)\,P\,[(c_2 + c_4)\,x + (c_1 - c_3)\,y + c_9\,u + c_{10}\,v] \ldots\ldots (30)$$

It is observed that the constants c_1, c_2, c_3, and c_4 have been written in Eq. 30 in such a way that the two equations of compatibility corresponding to Eqs. 25 will be satisfied. If Eqs. 29 and 30 are to be used directly, the origin of coordinates should be chosen so that x, y, u, and v can all be kept small.

When the load P is distributed over the area A, the term P in Eqs. 29 and 30 can be replaced by $p\,dA$, and thereafter the expressions on the right side can be integrated with respect to u and v over the area A. The resulting supplementary moments at point x,y will still be expressed by Eqs. 29 and 30 provided that u and v are now re-interpreted as the coordinates of the "center of the load," by which is meant the point of application of the resultant of the distributed load. It follows that there is an advantage in choosing the origin of the coordinates at the center of the load; namely, that the terms in Eqs. 29 and 30 containing u and v and the six constants c_5 to c_{10} thereby drop out. If, besides, the moments or stresses are computed at the center of the load, so that $x = y = 0$, the supplementary moments corresponding to the terms of third degree in the expression for the deflection disappear altogether. At other points the supplementary moments or stresses will be sums of terms pro-

portional to first moments of the load. Inclusion of terms of fourth degree in the expression for the deflection would produce further supplementary terms for the moments and stresses. These further terms would contain moments of inertia and products of inertia of the load.

A useful conclusion can be drawn from these deliberations: It will be desirable generally to determine the place coefficients B, C, and D so that they will be exact at or near the center of the load. When this is done, the origin of coordinates may again be chosen freely according to convenience; in the region of interest the supplementary terms that might otherwise be considered will remain small; and the formulas in Eqs. 13 to 20 may be used without supplementary terms.

Corrections needed when the loaded area is too large to be rated as small will be considered in a later section.

Determination of the Place Coefficients in Six Examples

Example a. Center of a Circular Plate or Slab with a Fixed Edge at $r = c$.—

When a load P is concentrated at the center, the deflection at any point can be written in the form of Eq. 1 as follows:[5, 6]

$$\zeta = \frac{P}{8\,\pi\,N}\left[r^2\left(\log\frac{r}{c} - \frac{1}{2}\right) + \frac{1}{2}\,c^2\right] \quad (31)$$

A comparison with Eq. 1 shows that with c chosen equal to the radius

$$B = -\frac{1}{2} \quad \text{and} \quad C = D = 0 \quad (32)$$

Example b. Center of a Circular Plate or Slab with a Simply Supported Edge at $r = c$.—

With the load P at the center the deflection at any point can again be written in the form of Eq. 1 with $R = r$ and $C = D = 0$. It follows that Eq. 8 for the moments applies over the whole area. Then at $y = 0$, m_x becomes

$$m_r = -\frac{(1+\mu)P}{4\,\pi}\log\frac{r}{c} \quad (33)$$

which after further comparison with Eq. 8 gives

$$B = \frac{1-\mu}{2(1+\mu)} \quad \text{and} \quad C = D = 0 \quad (34)$$

The values in Eq. 34 can also be obtained like those in Eq. 32 by inspection of the formula for the deflection, which is in this case[6, 7]

$$\zeta = \frac{P}{8\,\pi\,N}\left[r^2\log\frac{r}{c} + \frac{3+\mu}{2\,(1+\mu)}\,(c^2 - r^2)\right] \quad (35)$$

[5] See, for example, "Die elastischen Platten," by A. Nádai (Julius Springer, Berlin), 1925, p. 61; or footnote 6.

[6] "Theory of Plates and Shells," by S. Timoshenko (McGraw-Hill Book Company, Inc., New York, N. Y.), 1940, p. 74.

[7] "Die elastischen Platten," by A. Nádai (Julius Springer, Berlin), 1925, p. 60.

Example c. Long Rectangular Plate or Slab Extending Far in the Directions of $\pm y$, with Simply Supported Edges at $x = \pm \frac{1}{2} l$, and with a Load P at $x = u$, $y = 0$.—

A particularly usable solution of this problem was given by A. Nádai[8, 9] in 1921. According to this solution one may write

$$- N \Delta \zeta = \frac{P}{2\pi} \operatorname{Re} \left[\log \cos \frac{\pi (z + u)}{2l} - \log \sin \frac{\pi (z - u)}{2l} \right] \ldots\ldots (36)$$

in which z denotes the complex variable $x + i y$ and the symbol Re stands for "real part of the function." Furthermore

$$m_x = m_y \quad \text{at} \quad y = 0 \ldots\ldots (37)$$

When R, which is the same as the numerical value of $z - u$, is small, Eq. 36 may be written in the simpler form

$$- N \Delta \zeta = \frac{P}{2\pi} \left[\log \left(\frac{2}{\pi} \cos \frac{\pi u}{2l} \right) - \log \frac{R}{l} \right] \ldots\ldots (38)$$

It is convenient to choose $c = l$. A comparison of Eqs. 38 and 37 with Eqs. 4 and 8 then leads directly to the results

$$c = l, \qquad B = \log \left(\frac{2}{\pi} \cos \frac{\pi u}{2l} \right) = - 0.4516 + 2.3026 \log_{10} \cos \frac{\pi u}{2l},$$

$$C = 1 \quad \text{and} \quad D = 0 \ldots\ldots (39)$$

Example d. Special Case of Example c: The Load Is at Midspan, at $x = y = 0$.—

Eqs. 39 give

$$c = l, \qquad B = - 0.4516, \qquad C = 1, \quad \text{and} \quad D = 0 \ldots\ldots (40)$$

Example e. Again a Long Rectangular Plate or Slab Extending Far in the Directions of $\pm y$, but with Fixed Edges at $x = \pm \frac{1}{2} l$, with the Load P at Midspan at $x = y = 0$.—

A supplement[10] to Nádai's solution for the simply supported plate or slab gives results that can be stated as follows:

$$c = l, \qquad B = - 1.044, \qquad C = 0.538, \qquad \text{and} \qquad D = 0 \ldots\ldots (41)$$

Example f. Large Slab on Elastic Subgrade Offering a Reaction per Unit of Area Equal to the Deflection Times a Constant "Modulus of Subgrade Reaction" k; the Place of the Load Being Far from Any Edge.—

This problem was solved in 1884 by Heinrich Hertz[11] who considered a

[8] "Über die Spannungsverteilung in einer durch eine Einzelkraft belasteten rechteckigen Platte," by A. Nádai, *Der Bauingenieur*, Vol. 2, 1921, pp. 11–16; or "Die elastischen Platten," by A. Nádai (Julius Springer, Berlin), 1925, pp. 86 and 93.

[9] "Computation of Stresses in Bridge Slabs Due to Wheel Loads," by H. M. Westergaard, *Public Roads*, Vol. 11, March, 1930, pp. 1–23, especially p. 6.

[10] *Ibid.*, especially p. 20, formulas 104 and 105.

[11] "Über das Gleichgewicht schwimmender elastischer Platten," by Heinrich Hertz, *Wiedemanns Annalen der Physik und Chemie*, Vol. 22, 1884, pp. 449–455; reprinted in his "Gesammelte Werke," Vol. 1, pp. 288–294.

plate floating on a liquid of weight k per unit of volume, and loaded by a concentrated force P. The solution can be applied to problems of concrete pavements if it is recognized in advance that the assumption of the existence of a constant modulus k of subgrade reaction is at best a workable compromise in the interest of simplicity.[12,13,14]

A simple complete derivation will be shown.

The general equation of flexure of plates or slabs, Lagrange's equation

$$N \Delta^2 \zeta = p \qquad (42)$$

becomes in this case

$$N \Delta^2 \zeta + k \zeta = 0 \qquad (43)$$

When a distance l, known as the "radius of relative stiffness," and defined by the equation

$$k l^4 = N \qquad (44)$$

is introduced, Eq. 43 becomes

$$l^4 \Delta^2 \zeta + \zeta = 0 \qquad (45)$$

If the function

$$Z = \zeta + i l^2 \Delta \zeta \qquad (46)$$

(in which $i = \sqrt{-1}$) satisfies the equation

$$l^2 \Delta Z + i Z = 0 \qquad (47)$$

ζ will satisfy Eq. 45. Eq. 47 is satisfied[15] when Z is taken as a constant times any Bessel function of order zero with the argument $\frac{r\sqrt{i}}{l}$. Hankel's Bessel function $H^{(1)}{}_0$ is selected.[16] Then it will be in agreement with Eqs. 45 to 47 to write

$$\zeta = \frac{P l^2}{4 N} \operatorname{Re} H^{(1)}{}_0 \left(\frac{r\sqrt{i}}{l}\right) \qquad (48)$$

and

$$\Delta \zeta = \frac{P}{4 N} \operatorname{Im} H^{(1)}{}_0 \left(\frac{r\sqrt{i}}{l}\right) \qquad (49)$$

in which the symbol Re stands for the real part and the symbol Im for the imaginary part of the function. Eq. 48 is a statement of Hertz's solution in a particularly simple form.

To show that Eqs. 48 and 49 apply (without need for adding other particular solutions containing integration constants) in the case of a large slab with the load P at $r = 0$ three further observations will be sufficient: First, both ζ and $\Delta \zeta$ vanish when r goes to infinity; second, $d\zeta/dr = 0$ at $r = 0$; and third, when

[12] "Stresses in Concrete Pavements Computed by Theoretical Analysis," by H. M. Westergaard, *Public Roads*, Vol. 7, April, 1926, pp. 25–35.

[13] "Analytical Tools for Judging Results of Structural Tests of Concrete Pavements," by H. M. Westergaard, *ibid.*, Vol. 14, December, 1933, pp. 185–188.

[14] "Stresses in Concrete Runways of Airports," by H. M. Westergaard, *Proceedings*, Nineteenth Annual Meeting of the Highway Research Bd., December, 1939, published in 1940, pp. 197–202.

[15] "Funktionentafeln mit Formeln und Kurven," by E. Jahnke and F. Emde (B. G. Teubner, Leipzig), 1909, p. 166.

[16] *Ibid.*, pp. 97 and 139.

r is small, Eq. 49 may be written in the form[17]

$$N \Delta \zeta = \frac{P}{2\pi}\left(\log\frac{r}{l} - 0.1159\right) \ldots\ldots\ldots\ldots\ldots\ldots (50)$$

Eq. 50 is of the form of Eq. 4 and shows that the load P is accounted for.

A comparison of Eq. 50 with Eq. 4 and a consideration of the symmetry lead to the following statement of results for the slab on elastic subgrade:

$$c = l, \qquad B = 0.1159, \qquad \text{and} \qquad C = D = 0 \ldots\ldots\ldots\ldots (51)$$

with l defined by Eq. 44.

Statement of the Area Coefficients in Twelve Cases

The area coefficients K, S, and T which appear in Eqs. 13 to 20 are defined by Eqs. 21 and 22 when the load is distributed uniformly over the loaded area; otherwise they are defined by Eqs. 11 and 12. Some formulas will now be stated. Notes on their derivation follow later.

Case 1.—

Circle $r = a$ with the load distributed uniformly over the area.

Within the circle:

$$K = \frac{1}{2} + \log\frac{c}{a} - \frac{r^2}{2a^2}, \qquad S = -\frac{r^2}{2a^2}\cos 2\theta, \qquad T = -\frac{r^2}{2a^2}\sin 2\theta \ldots (52)$$

Outside the circle:

$$K = \log\frac{c}{r}, \qquad S = \left(\frac{a^2}{2r^2} - 1\right)\cos 2\theta, \qquad T = \left(\frac{a^2}{2r^2} - 1\right)\sin 2\theta \ldots (53)$$

Case 2.—

Narrow strip along the whole periphery of the circle $r = a$, with the load distributed uniformly over the length.

Inside the circle:

$$K = \log\frac{c}{a}, \qquad S = T = 0 \ldots\ldots\ldots\ldots\ldots\ldots (54)$$

Outside the circle:

$$K = \log\frac{c}{r}, \qquad \frac{S}{\cos 2\theta} = \frac{T}{\sin 2\theta} = \frac{a^2}{r^2} - 1 \ldots\ldots\ldots\ldots (55)$$

Case 3.—

Circle $r = a$ loaded over the area according to the formula

$$p = \frac{(n+2)\,P\,r^n}{2\pi\,a^{n+2}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (56)$$

in which $n > -2$.

Within the circle:

$$K = \frac{1}{n+2} + \log\frac{c}{a} - \frac{1}{n+2}\left(\frac{r}{a}\right)^{n+2} \ldots\ldots\ldots\ldots (57)$$

$$\frac{S}{\cos 2\theta} = \frac{T}{\sin 2\theta} = -\frac{2}{n+4}\left(\frac{r}{a}\right)^{n+2} \ldots\ldots\ldots\ldots (58)$$

[17] "Funktionentafeln mit Formeln und Kurven," by E. Jahnke and F. Emde (B. G. Teubner, Leipzig), 1909, pp. 97, 93, and 141.

Outside the circle:

$$K = \log \frac{c}{r}, \qquad \frac{S}{\cos 2\theta} = \frac{T}{\sin 2\theta} = \frac{n+2}{n+4}\left(\frac{a}{r}\right)^2 - 1 \quad\ldots\ldots\ldots(59)$$

It is observed that $n = 0$ reproduces Case 1 and $n = \infty$ Case 2.

Case 4.—

Circle $r = a$ loaded over the area according to the formula

$$p = \frac{(n+2)\,P}{n\,\pi\,a^2}\left[1 - \left(\frac{r}{a}\right)^n\right] \quad\ldots\ldots\ldots(60)$$

in which $n > -2$ and $n \neq 0$.

Within the circle:

$$K = \frac{n+4}{2\,(n+2)} + \log\frac{c}{a} - \frac{n+2}{2\,n}\left(\frac{r}{a}\right)^2 + \frac{2}{n\,(n+2)}\left(\frac{r}{a}\right)^{n+2} \quad\ldots\ldots(61)$$

$$\frac{S}{\cos 2\theta} = \frac{T}{\sin 2\theta} = -\frac{n+2}{2\,n}\left(\frac{r}{a}\right)^2 + \frac{4}{n\,(n+4)}\left(\frac{r}{a}\right)^{n+2} \quad\ldots\ldots(62)$$

Outside the circle:

$$K = \log\frac{c}{r}, \qquad \frac{S}{\cos 2\theta} = \frac{T}{\sin 2\theta} = \frac{n+2}{2\,(n+4)}\left(\frac{a}{r}\right)^2 - 1 \quad\ldots\ldots(63)$$

It is observed that $n = \infty$ reproduces Case 1.

Case 5.—

Ellipse

$$x^2/a^2 + y^2/b^2 = 1 \quad\ldots\ldots\ldots(64)$$

with the load distributed uniformly over the area.

Within the area of the ellipse:

$$K = \frac{1}{2} + \log\frac{2\,c}{a+b} - \frac{x^2}{a\,(a+b)} - \frac{y^2}{b\,(a+b)} \quad\ldots\ldots\ldots(65)$$

$$S = -\frac{a-b}{a+b} - \frac{2\,a\,b}{(a+b)^2}\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right) \quad\ldots\ldots\ldots(66)$$

$$T = -\frac{4\,x\,y}{(a+b)^2} \quad\ldots\ldots\ldots(67)$$

Outside the area of the ellipse it is expedient to use a function of the complex variable $z = x + i\,y$; namely,

$$W = \frac{z - \sqrt{z^2 - a^2 + b^2}}{a^2 - b^2} = \frac{1}{z + \sqrt{z^2 - a^2 + b^2}} \quad\ldots\ldots\ldots(68)$$

which at the elliptic boundary has the value

$$W_b = \frac{x}{a\,(a+b)} - \frac{i\,y}{b\,(a+b)} \quad\ldots\ldots\ldots(69)$$

and which may be written for great values of z in the form

$$W = \frac{1}{2\,z} + \frac{a^2 - b^2}{8\,z^3} + \cdots \qquad (70)$$

The following values, which are stated in terms of W in Eq. 68, apply outside the elliptic area and coincide with the values in Eqs. 65 to 67 at the boundary:

$$K = \frac{1}{2} + \text{Re}\,[\log\,(2\,c\,W) - z\,W] \qquad (71)$$

$$S = -\frac{a-b}{a+b} - \frac{4}{a+b}\,(b\,x\,\text{Re} + a\,y\,\text{Im})\,W + 2\,a\,b\,\text{Re}\,(W^2) \qquad (72)$$

$$T = \frac{4}{a+b}\,(b\,x\,\text{Im} - a\,y\,\text{Re})\,W - 2\,a\,b\,\text{Im}\,(W^2) \qquad (73)$$

in which Re stands for "the real part of" and Im for "the imaginary part of."

At $y = 0$, $x \geqq a$, Eqs. 71 to 73 give

$$K = \frac{1}{2} + \log\frac{2\,c}{x + \sqrt{x^2 - a^2 + b^2}} - \frac{x}{x + \sqrt{x^2 - a^2 + b^2}} \qquad (74)$$

$$S = -\frac{a^2 + b^2}{a^2 - b^2} + \frac{4\,b^2\,x}{(a^2 - b^2)\,(x + \sqrt{x^2 - a^2 + b^2})} \qquad (75)$$

$$T = 0 \qquad (76)$$

At $x = 0$, $y \geqq b$ one finds similarly

$$K = \frac{1}{2} + \log\frac{2\,c}{y + \sqrt{y^2 + a^2 - b^2}} - \frac{y}{y + \sqrt{y^2 + a^2 - b^2}} \qquad (77)$$

$$S = -\frac{a^2 + b^2}{a^2 - b^2} + \frac{4\,a^2\,y}{(a^2 - b^2)\,(y + \sqrt{y^2 + a^2 - b^2})} \qquad (78)$$

$$T = 0 \qquad (79)$$

When either y is great or b is fairly close to a, S in Eq. 78 will be close to the value

$$S' = 1 - \frac{a^2}{2\,y^2} + \frac{a^2\,(a^2 - b^2)}{4\,y^4} \qquad (80)$$

For example, with $a = 1.05\,b$, at $x = 0$, $y = b$, Eq. 80 gives $S' = 0.477$, whereas Eq. 78 or Eq. 66 gives $S = 0.475$. The first two terms in the expression in Eq. 80 are suggested by the formula for S in Eqs. 53, which apply when $a = b$.

Case 6.—

Narrow strip along the axis of x with the load $q = q(x)$ per unit of length; q being different from zero only over a short stretch, and

$$\int_{-\infty}^{\infty} q\,dx = P \qquad (81)$$

The solution may be written in terms of a function Z of the complex variable $z = x + i\,y$ in the form

$$K = \operatorname{Re} Z, \qquad S = 2\,y \operatorname{Im} Z' - 1, \qquad T = 2\,y \operatorname{Re} Z' \ldots\ldots (82)$$

in which Re stands for "the real part of" and Im for "the imaginary part of," and Z' denotes the first derivative with respect to z; provided that Z and Z' are determined so that they satisfy the following three conditions, the first two of which define Z':

First,

$$\operatorname{Im} Z' = \pm \frac{\pi\,q}{P} \quad \text{at} \quad y = 0 \ldots\ldots\ldots\ldots\ldots\ldots (83)$$

with the plus and minus signs applying when y approaches zero from the positive and negative side respectively.

Second, $z\,Z'$ must converge toward -1 when z goes to infinity.

Third, in obtaining Z from Z' the integration constant must be chosen so that Re Z converges toward log (c/r) when z goes to infinity.

Case 7.—

Narrow strip along the axis of x from $x = -a$ to $x = a$, with the load per unit of length equal to

$$q = \frac{2\,P}{\pi\,a^2} \sqrt{a^2 - x^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (84)$$

The solution may be stated either as an application of Case 5 or as an illustration of Case 6.

As application of Case 5: Eqs. 65 to 79 apply with $b = 0$.

As illustration of Case 6: Eqs. 82 apply with

$$Z' = -\frac{2}{a^2}\left(z - \sqrt{z^2 - a^2}\right) \ldots\ldots\ldots\ldots\ldots\ldots (85)$$

and

$$Z = \frac{1}{2} + \log \frac{2\,c}{z + \sqrt{z^2 - a^2}} - \frac{z}{z + \sqrt{z^2 - a^2}} \ldots\ldots\ldots\ldots (86)$$

Case 8.—

Narrow strip along the axis of x from $x = -a$ to $x = a$, loaded uniformly over the length.

Eqs. 82 for Case 6 apply with

$$Z' = \frac{1}{2\,a} \log \frac{z - a}{z + a} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (87)$$

$$Z = 1 + \frac{z - a}{2\,a} \log \frac{z - a}{c} - \frac{z + a}{2\,a} \log \frac{z + a}{c} \ldots\ldots\ldots\ldots (88)$$

Case 9.—

Narrow strip 1–3 in Fig. 1, loaded uniformly over the length.

Values at point 0:

$$K = 1 + \frac{s}{b} \log \frac{c}{R_1} + \frac{t}{b} \log \frac{c}{R_3} - \frac{R_2 \beta}{b} \quad \text{.............. (89)}$$

$$S = \left[1 - \frac{2 R_2 \beta}{b} \right] \cos 2 \alpha + \frac{2 R_2}{b} \left(\log \frac{R_3}{R_1} \right) \sin 2 \alpha \quad \text{....... (90)}$$

$$T = \left[1 - \frac{2 R_2 \beta}{b} \right] \sin 2 \alpha - \frac{2 R_2}{b} \left(\log \frac{R_3}{R_1} \right) \cos 2 \alpha \quad \text{....... (91)}$$

Case 10.—
Triangle 0–1–3 in Fig. 1, with the load distributed uniformly over the area A.

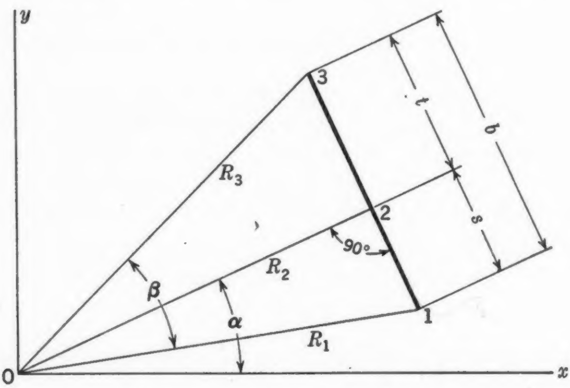


FIG. 1.—STRIP 1–3 IN CASE 9 AND TRIANGLE 0–1–3 IN CASE 10

Values at point 0: For K: add $\frac{1}{2}$ to the value in Eq. 89. The coefficients S and T are the same as in Eqs. 90 and 91.

Case 11.—
Polygon composed of triangles such as 0-($2n - 1$)-($2n + 1$) in Fig. 2, with $n = 1, 2, 3 \ldots$; the load being distributed uniformly over the whole area A.

Values at the common vertex 0:

$$K = \frac{3}{2} + \frac{1}{2A} \sum^{n} R_{2n} \left(s_n \log \frac{c}{R_{2n-1}} + t_n \log \frac{c}{R_{2n+1}} - R_{2n} \beta_n \right) \quad \text{... (92)}$$

$$S = \frac{1}{A} \sum^{n} R_{2n} \left[\left(\frac{1}{2} b_n - R_{2n} \beta_n \right) \cos 2 \alpha_n + R_{2n} \left(\log \frac{R_{2n+1}}{R_{2n-1}} \right) \sin 2 \alpha_n \right] \quad \text{................ (93)}$$

$$T = \frac{1}{A} \sum^{n} R_{2n} \left[\left(\frac{1}{2} b_n - R_{2n} \beta_n \right) \sin 2 \alpha_n - R_{2n} \left(\log \frac{R_{2n+1}}{R_{2n-1}} \right) \cos 2 \alpha_n \right] \quad \text{................ (94)}$$

Case 12.—
Rectangle with sides a and b in the directions of x and y, loaded uniformly over the area.

At the center:

$$K = \frac{1}{2}\left[3 + \log \frac{4\,c^2}{a^2 + b^2} - \frac{a}{b}\tan^{-1}\left(\frac{b}{a}\right) - \frac{b}{a}\tan^{-1}\left(\frac{a}{b}\right)\right] \qquad (95)$$

$$S = -\frac{a}{b}\tan^{-1}\left(\frac{b}{a}\right) + \frac{b}{a}\tan^{-1}\left(\frac{a}{b}\right) \qquad (96)$$

$$T = 0 \qquad (97)$$

At the two corners at which both x and y have either their smallest values or their greatest values:

$$K = K_{\text{Eq. 95}} - \log 2 \qquad (98)$$

Coefficient S is the same as in Eq. 96.

$$T = \frac{a}{2\,b}\log \frac{a^2}{a^2 + b^2} + \frac{b}{2\,a}\log \frac{b^2}{a^2 + b^2} \qquad (99)$$

At the center of either side of length b: K, S, and T are the same as in Eqs. 95 to 97 except that a is replaced by $2\,a$.

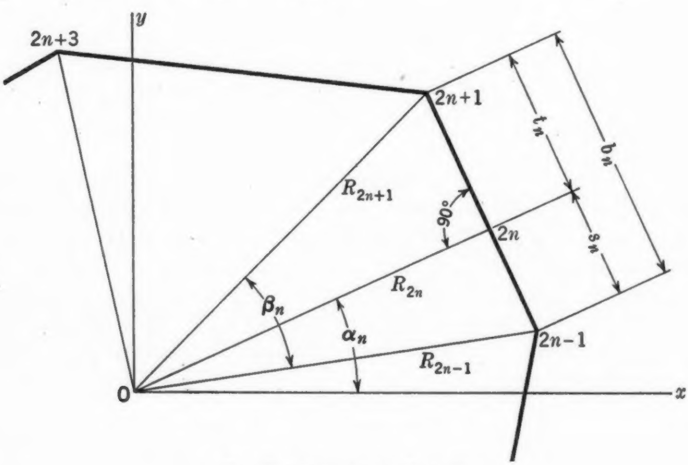


FIG. 2.—POLYGON IN CASE 11

NOTES ON THE DERIVATION OF THE PRECEDING FORMULAS FOR AREA COEFFICIENTS

Some of the formulas for the area coefficients can be derived conveniently by direct use of the definitions in Eqs. 11 and 12, or 21 and 22. For example, in this way the formulas for the uniformly loaded circular area in Case 1, Eqs. 52 and 53, can be obtained by easy integrations; and the formulas for the interior of the uniformly loaded elliptic area in Case 5, Eqs. 65 to 67, can be

obtained by less simple, yet manageable integrations. (Indebtedness is expressed to Louis A. Pipes, Faculty Instructor in Electrical Engineering, Harvard University, who derived Eq. 65 independently and commented on electrical analogies that exist because K is a logarithmic potential.)

In most of the cases, however, including that of the elliptic area in Case 5, if values are desired at all points, it is more convenient to use the governing differential equations.

In Cases 1 to 4 the load is confined by the circle $r = a$; p is a function of r only, and $p = 0$ when $r > a$. The governing differential equations are Eq. 27,

$$\frac{d^2K}{dr^2} + \frac{1}{r}\frac{dK}{dr} = -\frac{2\pi p}{P} \qquad (100)$$

and Eq. 28,

$$\frac{df}{dr} + \frac{2f}{r} = 2\frac{dK}{dr} \qquad (101)$$

in which, by Eqs. 26,

$$f = \frac{S}{\cos 2\theta} = \frac{T}{\sin 2\theta} \qquad (102)$$

If these equations are to be used, it is necessary to note some conditions that apply at special places. The following conditions of this kind are consequences of the definitions of K, S, and T in Eqs. 11 and 12: First, when r goes to infinity, K must converge toward or be equal to $\log(c/r)$ and f must converge toward -1; second, K and f must be continuous functions, so that they suffer no sudden change by crossing the boundary $r = a$; third, dK/dr must be continuous except possibly at places of concentration of load, as at $r = a$ in Case 2, or at $r = 0$ in Cases 3 and 4 if n is negative; and fourth, $f = 0$ at $r = 0$.

In Cases 1 to 4 the only possible form of K for $r \geqq a$ that satisfies both Eq. 100 and the special condition for $r = \infty$ is $K = \log(c/r)$. This gives $K = \log(c/a)$ at $r = a$. Thereafter the only possible forms of K for $r \leqq a$ are found to be those stated in Eqs. 52, 54, 57, and 61, respectively; it being noted in Cases 1, 3, and 4 that dK/dr is continuous across the boundary $r = a$. With $f = 0$ at $r = 0$ the only possible forms of f for $r \leqq a$ are those defined by the values of S and T in Eqs. 52, 54, 58, and 62, respectively. For $r > a$ the only possible form of f is

$$f = \frac{\text{constant}}{r^2} - 1 \qquad (103)$$

in which the constant can be determined by the value of f at $r = a$; this leads to the formulas for S and T in Eqs. 53, 55, 59, and 63. For supplementary checking it may be observed that df/dr is continuous across the boundary at $r = a$ except in Case 2, in which the load is concentrated at the boundary.

In the less simple cases in which p is not a function of r only the differential equations governing K, S, and T are Eq. 24,

$$\Delta K = -\frac{2\pi p}{P} \qquad (104)$$

and Eqs. 25,

$$\frac{\partial}{\partial x}(2\,K - S) = \frac{\partial T}{\partial y}, \qquad \frac{\partial}{\partial y}(2\,K + S) = \frac{\partial T}{\partial x} \quad \ldots\ldots\ldots (105)$$

Three conditions which apply at special places, and which are consequences of the definitions of K, S, and T in Eqs. 11 and 12, will make the solutions unique. The first is that when r goes to infinity, K, S, and T must converge toward the values

$$K_\infty = \log\frac{c}{r}, \qquad S_\infty = -\cos 2\,\theta, \qquad T_\infty = -\sin 2\,\theta \quad \ldots\ldots (106)$$

The second and third conditions will be stated under the assumption that p^2 has a finite maximum value except on lines where load may be concentrated; on such lines the load q per unit of length must nowhere be infinite; and no finite part of the load may be concentrated at a point. Then the second condition is that K, S, and T must be continuous functions with single values, finite or zero, at all points of the xy-plane except at $r = \infty$. The third condition is that $\partial K/\partial x$ and $\partial K/\partial y$ must nowhere be infinite and must be single-valued at all points except at lines where the load is concentrated as a load q per unit of length of the line; in crossing such a line in the direction of a rectangular coordinate n along the normal, the value of $\partial K/\partial n$ is required to drop by the amount $2\,\pi\, q/P$, giving the surface x,y,K a corresponding edge or sharp ridge.

When the functions K, S, and T satisfy the second and third special conditions and Eqs. 105, the first derivatives of S and T with respect to x and y will automatically be single-valued at all points except at lines where load is concentrated. The surfaces x,y,S and x,y,T can have no sharp edge except where the surface x,y,K also has one. This conclusion may be used for supplementary verification of results, but the three conditions that were stated and Eqs. 104 and 105 are sufficient to define the single solution that exists in each case.

Three additional, general observations are needed. The first is that a general solution of Eqs. 104 and 105 for the unloaded region, in which $p = 0$, may be written in the following form:

$$K = \mathrm{Re}\, Z \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (107)$$

$$S = 2\,j\,x\,\mathrm{Re}\,Z' + 2\,(1 - j)\,y\,\mathrm{Im}\,Z' + \mathrm{Re}\,Z_1 \quad \ldots\ldots\ldots (108)$$

$$T = -2\,j\,x\,\mathrm{Im}\,Z' + 2\,(1 - j)\,y\,\mathrm{Re}\,Z' - \mathrm{Im}\,Z_1 \quad \ldots\ldots\ldots (109)$$

in which Re stands for "the real part of" and Im for "the imaginary part of," and Z and Z_1 are any analytic functions of the complex variable z, and j is a convenient constant.

The second general observation is that if the loaded area is turned about the point u,v through an angle α_0, so that the original angles α are changed to $\alpha + \alpha_0$, the value of K at point u,v will remain unchanged; but the values S_0 and T_0 of S and T at point u,v will be changed, according to the definitions

in Eqs. 12, to

$$S = -\frac{1}{P}\int dP \cos 2\,(\alpha + \alpha_0) = S_0 \cos 2\,\alpha_0 - T_0 \sin 2\,\alpha_0 \ldots\ldots (110)$$

$$T = -\frac{1}{P}\int dP \sin 2\,(\alpha + \alpha_0) = S_0 \sin 2\,\alpha_0 + T_0 \cos 2\,\alpha_0 \ldots\ldots (111)$$

The third general observation deals with the effects of an expansion of the loaded area. Consider three areas A_1, A_2, and A_3 over each of which the load P is distributed uniformly in succession. The area A_1 has the boundary $r = r_1\,(\theta)$; A_2 has the boundary $r = (1 + j)\,r_1\,(\theta)$, in which j is a positive constant; and A_3 is the part of A_2 lying outside A_1. At $r = 0$, S and T will have the same values for the three areas; but K will have three values K_1, K_2, and K_3 at $r = 0$ for the three areas. The following relations apply:

$$K_2 = K_1 - \log\,(1 + j) \ldots\ldots (112)$$

$$A_3\,K_3 = A_2\,K_2 - A_1\,K_1 \ldots\ldots (113)$$

$$A_2 = (1 + j)^2\,A_1, \qquad A_3 = j\,(2 + j)\,A_1 \ldots\ldots (114)$$

Substitutions from Eqs. 112 and 114 in Eq. 113 give

$$K_3 = K_1 - \frac{(1 + j)^2 \log\,(1 + j)}{j\,(2 + j)} \ldots\ldots (115)$$

When j is very small, so that A_3 is reduced to a rim around A_1, K_3 in Eq. 115 becomes

$$K_4 = K_1 - \frac{1}{2} \ldots\ldots (116)$$

The preparations have now been made for the discussion of the remaining individual cases.

In Case 5 an elliptic area is loaded uniformly. The solution in Eqs. 65 to 67 and 71 to 73 is verified by the following observations: First, Eqs. 72 and 73 fit the pattern of Eqs. 108 and 109 when W is any analytic function of the complex variable z; but when W is chosen as in Eq. 68, all three Eqs. 71 to 73 fit the pattern of Eqs. 107 to 109. Second, with W as in Eq. 68, the values in Eqs. 71 to 73 converge toward the values in Eqs. 106 when r goes to infinity, and assume the values defined by Eqs. 65 to 67 on the elliptic boundary. Third, the values of K, S, and T in Eqs. 65 to 67 satisfy Eqs. 104 and 105. Fourth and last, the first derivatives of K with respect to x or y are the same at the elliptic boundary whether they are determined from Eq. 65 or Eq. 71. The solution satisfies all the requirements and is correct. For an additional check it can be observed that the first derivatives of S and T with respect to x or y are the same at the elliptic boundary whether they are determined from Eqs. 66 and 67 or from Eqs. 72 and 73.

In Case 6 the load is at $y = 0$ only. Eqs. 82 fit the pattern of Eqs. 107 to 109. The requirements in Eqs. 106 for $r = \infty$ are satisfied; and Eq. 83 assures that in crossing the axis of x from the side of $- y$ to the side of y the value of

$\partial K/\partial y$, which is equal to $-\operatorname{Im} Z'$, drops suddenly by the amount $2\pi q/P$, thereby accounting for the load q per unit of length. All requirements are satisfied. The solution is of the type introduced by Nádai[8] in his paper of 1921 on concentrated loads.

Case 7 is an obvious application of Case 5, obtained by letting one principal radius b of the ellipse approach zero; but Case 7 is much simpler than Case 5 and is solved conveniently by direct application of Case 6. The solution of Case 7 serves therefore as an additional check on the solution of Case 5.

Case 8, in which the load is distributed uniformly over the length of a piece of line, is a direct application of Case 6.

In Case 9 the load is distributed uniformly over the length of the line 1–3 in Fig. 1. The area coefficients are determined only at the origin of the coordinates. The value of K may be obtained either through Case 8 or by integration according to the definition in Eq. 11. The values of S and T are derived conveniently by determining first their values $-S_0$ and $-T_0$ when α in Fig. 1 is $-\pi/2$; this can be done either through Case 8 or by integrations according to the definitions in Eqs. 12. Thereafter Eqs. 110 and 111 are applied with α_0 equal to α in Fig. 1.

In Case 10 the load is distributed uniformly over the area of the triangle 0-1-3 in Fig. 1. The area coefficients are determined at 0 only. The coefficient K is obtained from Case 9 through Eq. 116; S and T are necessarily the same as in Case 9.

Case 11 of a polygon and Case 12 of a rectangle are direct applications of Case 10.

Three Examples of Corrections Needed when the Loaded Area Is Too Large To Be Rated as Small

Example 1.—Consider a circular plate or slab with a fixed edge at $r = c$ and with the load uniformly distributed over the area of the small circle $r = a$. Combination of the place coefficients in Eqs. 32, Example a, with the area coefficients in Eqs. 52, Case 1, gives at the center

$$B + K = \log \frac{c}{a}, \qquad C + S = D + T = 0 \tag{117}$$

These sums define the bending moment and the greatest stress at the center through Eqs. 13 and 15. The values are proportional to $B + K$.

When the radius a of the loaded area is increased so that it ceases to be small, a correction is needed. The correction can be made by replacing $B + K$ by a corrected combined coefficient $(B + K)'$, which in this example may be determined as follows:

According to Eqs. 4 and 32 a load $P = 1$ at the distance r from the center produces a value of $N \Delta \zeta$ at the center equal to

$$N(\Delta \zeta)_0 = \frac{1}{2\pi}\left[\log \frac{r}{c} + \frac{1}{2} + F(r)\right] \tag{118}$$

in which the function $F(r)$ is zero at $r = 0$ and contributes no singularities.

The surface $x, y, N\,(\Delta\,\zeta)_0$ is an influence surface and may therefore be produced as a deflected surface; it must obey the equation $\Delta^2\,(N\,\Delta\,\zeta)_0 = 0$ for $0 < r \leqq c$ and satisfy the conditions of the fixed edge at $r = c$. The only form of Eq. 118 that meets all these requirements is

$$N\,(\Delta\,\zeta)_0 = \frac{1}{2\,\pi}\left(\log\frac{r}{c} + \frac{1}{2} - \frac{r^2}{2\,c^2}\right)\ldots\ldots\ldots\ldots\ldots(119)$$

When the load P is distributed uniformly over the area of the circle $r = a$ the value of $N\,\Delta\,\zeta$ at the center becomes, by use of Eq. 119,

$$N\,\Delta\,\zeta = \frac{P}{\pi\,a^2}\int_0^a (2\,\pi\,r\,dr)\,N\,(\Delta\,\zeta)_0 = -\frac{P}{2\,\pi}\left(\log\frac{c}{a} + \frac{a^2}{4\,c^2}\right)\ldots(120)$$

According to Eq. 17 the uncorrected value of $N\,\Delta\,\zeta$ is $-\dfrac{P}{2\,\pi}\,(B + K)$. A comparison with Eq. 120 shows that the corrected value of $B + K$, which will replace the value in Eqs. 117, is

$$(B + K)' = \log\frac{c}{a} + \frac{a^2}{4\,c^2}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(121)$$

The last term in Eq. 121 represents the correction. When $a = c/4$, the correction is 1.1%. When $a = c$, the correction accounts for the whole value of the bending moment at the center.

Example 2.—The same plate or slab and the same load are considered as in Example 1 except that the edge at $r = c$ is now simply supported. Using Eqs. 34 instead of Eqs. 32 one finds

$$B + K = \log\frac{c}{a} + \frac{1}{1 + \mu},\qquad C + S = D + T = 0\ldots\ldots(122)$$

A corrected value $(B + K)'$ applicable when the radius a of the loaded circle ceases to be small can be obtained in the same way as in Example 1. Instead of Eqs. 119 and 121 one finds

$$N\,(\Delta\,\zeta)_0 = \frac{1}{2\,\pi}\left[\log\frac{r}{c} - \frac{1 - \mu}{2\,(1 + \mu)}\left(1 - \frac{r^2}{c^2}\right)\right]\ldots\ldots\ldots(123)$$

and

$$(B + K)' = \log\frac{c}{a} + \frac{1}{1 + \mu} - \frac{1 - \mu}{4\,(1 + \mu)}\,\frac{a^2}{c^2}\ldots\ldots\ldots\ldots(124)$$

The last term in Eq. 124 represents the correction. When $a = c/4$ and $\mu = 0.3$, the correction is $-$ 0.4%.

Example 3.—Consider a large slab on elastic subgrade under a load distributed uniformly over the area of a small circle $r = a$. Combination of the place coefficients in Example *f*, Eqs. 51, with the area coefficients in Case 1, Eqs. 52, gives at the center of the circle the sums

$$B + K = \log\frac{l}{a} + 0.6159,\qquad C + S = D + T = 0\ldots\ldots(125)$$

in which l is the radius of relative stiffness defined by Eq. 44. Eqs. 125 define the moments and stresses at the center of the circle through Eqs. 13 and 15. The values are proportional to $B + K$, as in Examples 1 and 2.

An analysis[18] applicable when the radius a has ceased to be small leads to a simple formula which will include the needed correction with good accuracy as long as $a < l$. As in Examples 1 and 2, the formula defines a corrected combined coefficient $(B + K)'$ which will replace $B + K$. The formula is

$$(B + K)' = \log \frac{l}{a} + 0.6159 + \frac{\pi}{32}\left(\frac{a}{l}\right)^2 \qquad (126)$$

The last term represents the correction. With $a = l/2$ the correction is 1.9%, which is still small. With $a = l/4$ the correction is 0.3%, which can well be ignored in practical applications.

Corrections Needed when the Loaded Area Is Very Small

When the load is concentrated within an area of very small extension, the usual assumption that horizontal normal stresses are proportional to the distance from a neutral plane will not apply in the immediate vicinity of the load. Results found by the ordinary theory of bending of plates or slabs should then be corrected in accordance with the theory of elasticity in three dimensions. When the corrections can be evaluated, and when the stresses at the bottom of the plate or slab are of primary interest, it will be suitable to state corrected results in terms of "substitute coefficients" K', S', and T'. These substitute coefficients should be determined so that they will define the correct stresses at the bottom of the plate or slab when they are used in Eqs. 15 and 16 instead of the true area coefficients K, S, and T.

When the load is concentrated at a point, the function K has an infinitely high peak at that point, while K' will have a rounded peak of finite height; this suggests the importance of replacing K by K' when the loaded area is reduced nearly to a point. On the other hand, S and T have the extreme values ± 1; which suggests that the replacements of S and T by S' and T' will usually be relatively unimportant, and may be ignored in most practical problems. The discussion that follows deals with the replacement of K only.

In Case 1, in which the load is distributed uniformly over the area of the circle $r = a$, the value of K at the center, according to Eqs. 52, is

$$K = \frac{1}{2} + \log \frac{c}{a} \qquad (127)$$

The value of the substitute coefficient K', which will define the correct stress at the bottom of the plate or slab under the center of the circle, may be written in the form

$$K' = \frac{1}{2} + \log \frac{c}{a'} \qquad (128)$$

in which a' is a substitute radius. Computations based on a theory given by

[18] "Stresses in Concrete Runways of Airports," by H. M. Westergaard, *Proceedings*, Nineteenth Annual Meeting of the Highway Research Bd., December, 1939, published in 1940, especially p. 202.

Nádai[19] led to the numerical values in Table 1 and to the following formulas which fit the numerical values with good approximation:[20]

$$a' = \sqrt{1.6\, a^2 + h^2} - 0.675\, h \quad \text{when} \quad a < 1.724\, h \qquad (129)$$

$$a' = a \quad \text{when} \quad a > 1.724\, h \qquad (130)$$

in which h is the thickness of the plate or slab.

If the load is distributed uniformly over a non-circular area, or if the area is circular but a point other than the center is considered, Eqs. 128 to 130, or Eq. 128 and Table 1, may still be used to obtain a proper substitute coefficient K'. If the circular area that was considered is replaced by the sector $\theta_1 \leqq \theta \leqq \theta_2$, $r \leqq a$, the value of K' at $r = 0$ will be the same as for the full circle; it follows that if the constant radius a is replaced by a single-valued function

$$a = a(\theta) \quad \text{for} \quad 0 \leqq \theta < 2\pi \qquad (131)$$

defining the boundary of a new area in polar coordinates a, θ, the new value of K' at $r = 0$ will be

$$K' = \frac{1}{2A} \int_0^{2\pi} a^2\, d\theta \left(\frac{1}{2} + \log \frac{c}{a'} \right) \qquad (132)$$

in which a' is the same function of a as before.

TABLE 1.—VALUES OF THE RATIO OF THE SUBSTITUTE RADIUS a' OF A UNIFORMLY LOADED CIRCULAR AREA TO THE THICKNESS h, FOR DIFFERENT VALUES OF THE RATIO OF THE TRUE RADIUS a TO THE THICKNESS h

a/h	a'/h
0	0.3254
0.25	0.3709
0.5	0.504
0.75	0.705
1.0	0.944
1.5	1.456
2.0	1.967

Assume next that the range $a_{max} - a_{min}$ of a in Eq. 131 is fairly small compared with the thickness h—for example, not greater than $h/4$—then within that range a' may be stated with good approximation by the formula

$$a' = a'_0 \left[1 + \lambda (a^2 - a^2_0)\right] \qquad (133)$$

in which a_0 is chosen so that

$$a^2_0 \int_0^{2\pi} a^2\, d\theta = \int_0^{2\pi} a^4\, d\theta \qquad (134)$$

and a'_0 is the value of a' for $a = a_0$, while λ is a suitable constant. According to Eq. 134 a_0 may be defined as $\sqrt{2}$ times the polar radius of gyration of the area A about $r = 0$, or as the radius of a circular area having the same polar radius of gyration about the center that A has about $r = 0$. The use of Eq. 133 means that a piece of the curve $a' = a'(a)$ is replaced by a suitable parabola. Eq. 133 may be restated approximately in the form

$$\log a' = \log a'_0 + \lambda (a^2 - a^2_0) \qquad (135)$$

[19] "Die Biegungsbeanspruchung von Platten durch Einzelkräfte," by A. Nádai, *Schweizerische Bauzeitung*, Vol. 76, 1920, p. 257; or "Die elastischen Platten," by A. Nádai (Julius Springer, Berlin), 1925, pp. 315–322.

[20] "Stresses in Concrete Pavements Computed by Theoretical Analysis," by H. M. Westergaard, *Public Roads*, Vol. 7, April, 1926, especially pp. 32, 26, and 27.

Substitution from Eqs. 135 and 134 in Eq. 132 then leads to the approximate formula

$$K' = \frac{1}{2} + \log \frac{c}{a'_0} \qquad (136)$$

which has the same form as Eq. 128 for the circular area. It is concluded that the value of K' can be found by first changing the loaded area into a circular area with the same polar radius of gyration, provided that the range $a_{max} - a_{min}$ is not too great.

When the load is distributed uniformly over the area of an ellipse with the principal radii a and b, as in Case 5, the value of K at the center is according to Eq. 65

$$K = \frac{1}{2} + \log \frac{2\,c}{a + b} \qquad (137)$$

The polar radius of gyration is proportional to $\sqrt{a^2 + b^2}$. Consequently, when the ellipse is either small enough or sufficiently close to a circle to permit the use of Eq. 136, the rule for changing K into K' may be stated as follows: The value of $\frac{1}{2}(a + b)$ in Eq. 137 is replaced according to Eq. 129 by the substitute radius

$$a'_0 = \sqrt{0.8\,(a^2 + b^2) + h^2} - 0.675\,h \qquad (138)$$

which can be used through Eq. 136. Eq. 129 should not be applied when $|a - b| > h/2$, nor when $a + b > 3\,h$. When either $|a - b| > h/2$ or $a + b > 2\,h$, it is suggested that instead of using Eq. 138, K' at the center be determined as if the ellipse were a circle with the radius $\frac{1}{2}(a + b)$; this means that a in Eqs. 129 and 130 should be replaced by $\frac{1}{2}(a + b)$. Such a computation gives stresses that are slightly too high for some of the combinations of a and b; but the errors are on the side of safety and disappear when a and b become large enough to permit the assumption that $K' = K$.

Numerical Example: Wheel with Dual Tires on Concrete Pavement

The numerical data defining this example are stated at the head of Table 2. Information kindly supplied by E. F. Kelley, Chief of the Division of Tests, Public Roads Administration, Federal Works Agency, in a letter, indicates that the assumptions stated at the head of Table 2 about the area of contact between tires and pavement represent with fair realism a case of a not excessive overload, such as may occur in a dynamic action of the rear wheels of a truck. The actual area of contact between a tire and the pavement is likely to be something between an ellipse and a circumscribed rectangle, but the difference between the assumed ellipses and a corresponding true area is not important from the present point of view.

With the numerical data stated at the head of Table 2, Eqs. 2, 44, and 51 give

$$N = 89{,}320{,}000 \text{ lb-in.}, \qquad l = c = 36.56 \text{ in.} \qquad (139)$$

$$B = 0.1159, \qquad C = 0 \qquad (140)$$

It is desirable to investigate first the effects of the load $P = 5{,}000$ lb transmitted through one of the tires. With this value of the load Eq. 15 for the normal stresses becomes

$$\left.\begin{matrix}\sigma_x \\ \sigma_y\end{matrix}\right\} = [58.465\,(B + K) \pm 19.488\,S]\ \text{lb-in.}^{-2} \dots\dots\dots (141)$$

but the possible necessity of replacing K by K' should be investigated. With $a = 3.25$ in. and $b = 6.5$ in. one finds at the center of the single loaded ellipse

TABLE 2.—NUMERICAL EXAMPLE OF STRESSES CREATED AT THE BOTTOM OF A SEVEN-INCH CONCRETE PAVEMENT BY A WHEEL LOAD OF 5,000 LB TRANSMITTED THROUGH A SINGLE TIRE OR BY A WHEEL LOAD OF 10,000 LB TRANSMITTED THROUGH DUAL TIRES, WHEN THE LOAD IS AT A CONSIDERABLE DISTANCE FROM ANY EDGE OR JOINT

Assumed modulus of elasticity of concrete, $E = 3{,}000{,}000$ lb-in.$^{-2}$ Poisson's ratio of the concrete, $\mu = 0.2$. Modulus of subgrade reaction, $k = 50$ lb-in.$^{-3}$

The load is assumed uniformly distributed over the area of contact between tires and pavement. In the case of the single tire the area of contact is bounded by the ellipse

$$\frac{x^2}{(3.25\text{ in.})^2} + \frac{y^2}{(6.5\text{ in.})^2} = 1 \dots\dots\dots (\text{Ellipse I})$$

In the case of the dual tires a second area is added; namely, the area bounded by a second ellipse

$$\frac{(x - 11.5\text{ in.})^2}{(3.25\text{ in.})^2} + \frac{y^2}{(6.5\text{ in.})^2} = 1 \dots\dots\dots (\text{Ellipse II})$$

The values are computed from Eqs. 141, 145, 146, and 147. The decimals are included to show the variations clearly.

AT DISTANCE x FROM CENTER OF ELLIPSE I		STRESSES (LB-IN.$^{-2}$) AT $y = 0$			
		Single Tire		Dual Tires	
x (inches)	Location	σ_x	σ_y	σ_x	σ_y
−3.25	Edge, ellipse I	132.15	136.47	173.25	213.01
0	Center, ellipse I	160.30	147.30	216.34	236.74
1.625	In ellipse I	153.26	144.59	218.52	241.53
3.25	Edge, ellipse I	132.15	136.47	208.35	241.75
4.50	Between ellipses	113.17	128.11	199.43	240.46
5.75	Midpoint	98.31	119.98	196.62	239.96
7.00	Between ellipses	86.26	112.35	199.43	240.46
8.25	Edge, ellipse II	76.20	105.28	208.35	241.75
9.875	In ellipse II	65.26	96.94	218.52	241.53
11.50	Center, ellipse II	56.04	89.44	216.34	236.74
14.75	Edge, ellipse II	41.10	76.54	173.25	213.01

by Eq. 138: $a'_0 = 4.827$ in.; by Eqs. 136, 139, and 140: $B + K' = 2.6406$; by Eq. 66: $S = \frac{1}{3}$; and finally by Eqs. 141 with K replaced by K' the stresses at the center:

$$\sigma'_x = 160.9\ \text{lb-in.}^{-2}, \qquad \sigma'_y = 147.9\ \text{lb-in.}^{-2} \dots\dots\dots (142)$$

When K is not replaced by K', one finds at the center by Eqs. 65, 139, and 140

$$B + K = 2.6307 \dots\dots\dots (143)$$

and thereafter by Eq. 141 the stresses at the center

$$\sigma_x = 160.30\ \text{lb-in.}^{-2}, \qquad \sigma_y = 147.30\ \text{lb-in.}^{-2} \dots\dots\dots (144)$$

which differ only insignificantly from the stresses in Eqs. 142. In the remaining computations K will not be replaced by K'.

When only ellipse I—defined at the head of Table 2—is loaded, the following values apply at points on the axis of x:

Within the ellipse, by Eqs. 143, 65, and 66:

$$B + K = 2.6307 - \frac{1}{3}\left(\frac{x}{a}\right)^2, \qquad S = \frac{1}{3} - \frac{4}{9}\left(\frac{x}{a}\right)^2 \qquad (145)$$

Outside the ellipse, by Eqs. 139, 140, 74, and 75, and with the notation

$$X = x + \sqrt{x^2 + 31.688 \text{ in.}^2} \qquad (146)$$

the corresponding values are

$$B + K = 0.6159 + \log \frac{73.12 \text{ in.}}{X} - \frac{x}{X}, \qquad S = \frac{5}{3} - \frac{16\,x}{3\,X} \qquad (147)$$

The stresses stated in the third and fourth columns of Table 2 were computed from Eqs. 141 and 145 to 147.

When the load $2\,P = 10{,}000$ lb is transmitted through two tires and is distributed uniformly over the area of the two ellipses defined at the head of Table 2, values of the stresses may be obtained by superposition. Since the distance between the centers of the two ellipses is 11.5 in., the stress σ_x or σ_y at the point x,0 due to the combined load $2\,P$ is the sum of two stresses produced by the load P over the area of ellipse I; namely, the stress at the point x, 0 plus the stress at the point (11.5 in. $- x$),0. The last two columns in Table 2 were obtained in this way by simple additions.

Conclusion

A pressure concentrated within a small area on the surface of a plate or slab produces a concentration of stress within and around that area. If the loaded area is not too close to any line or point of concentrated support nor to any edge, and if the main dimensions of the loaded area are not too small compared with the thickness of the plate or slab, and if the material of the plate or slab can be considered as homogeneous, isotropic, and elastic, the stresses and deformations in the region of concentration can be expressed by simple formulas in terms of a set of three "place coefficients" and a set of three "area coefficients." Each problem of this kind of stress concentration can be considered as solved when the needed place coefficients and area coefficients have been determined.

The place coefficients depend on the size and shape of the plate or slab, on the way in which the plate or slab is supported, and on the locality of the load, but do not depend on the size and shape of the loaded area, nor on the distribution of the load over that area. The area coefficients, on the other hand, depend only on the size and shape of the loaded area and on the manner of distribution of the load over that area. Of the three place coefficients or area coefficients in a set one is frequently zero for reasons of symmetry.

The number of problems for which solutions will have been made available at a given time by the method of place coefficients and area coefficients is the product of two numbers; namely, the number of available sets of place coefficients times the number of available sets of area coefficients.

Sets of place coefficients were derived in six examples by inspection of known solutions of particular problems in which the load is concentrated at a single point. The number of such examples can be increased without difficulty, because solutions for a load concentrated at a point are known in more cases than the six that were considered; and new problems of this kind can be solved by the methods of the theory of plates or slabs. Such new solutions will have an extended usefulness because place coefficients can be derived from them, and each new set of place coefficients can be combined with any set of area coefficients.

Area coefficients were derived in twelve cases. The areas confining the load are a circle in Cases 1 to 4, an ellipse in Case 5, narrow strips in Cases 6 to 9, a triangle in Case 10, any polygon in Case 11, and a rectangle in Case 12. While the range of these cases is representative, it will be desirable to have further cases added to the list.

Special devices of analysis are needed in the particular class of problems in which the main dimensions of the loaded area are small compared with the thickness of the plate or slab. Certain "substitute coefficients" were shown to be useful devices serving this purpose.

The use of place coefficients and area coefficients was illustrated by a numerical example. In this example the slab is a pavement, and the load was assumed to be distributed uniformly over the area of two ellipses which represent the area of contact between dual tires and the pavement. The stresses were computed by superposition of the effects identified with each of the two ellipses.

It has been shown that place coefficients and area coefficients can be used with advantage in solving many problems of stress concentrations in plates.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

PAPERS

PROFILE CURVES FOR OPEN-CHANNEL FLOW

BY DWIGHT F. GUNDER,[1] ESQ.

SYNOPSIS

Certain irregularities appearing in the surface profile curves for gradually varied flow are discussed in this paper. These irregularities occur under conditions in which the depth of flow is less than both the normal and critical depths. Further, the equation of gradually varied flow in a wide rectangular channel is integrated for the case where the Chézy coefficient is given by the Manning relation $C = 1.486 \frac{R^{1/6}}{n}$. As a consequence of the contents of this paper, the curves for a variable Chézy coefficient given by the Manning relation are sketched, and it is concluded that:

(1) Perhaps the backwater curves in most texts are based on certain qualitative features of the equation of gradually varied flow, plus a knowledge on the part of the writers of what these curves should look like;

(2) The use of the differential equation of gradually varied flow at depths below both the critical and normal depths should be restricted, in that the Manning relation for the Chézy coefficient should not be used, although in most cases, from a theoretical standpoint, the Bazin or Ganguillet-Kutter relations are satisfactory; and

(3) The differential equation of gradually varied flow with the Chézy coefficient given by the Manning relation is integrated for the case of the broad rectangular channel.

Notation.—The letter symbols in this paper are defined where they first appear in the text and are assembled for convenience of reference in the Appendix.

In calculating surface profile curves for gradually varied flow in open channels, using the customary step-by-step method in combination with the

NOTE.—Written comments are invited for immediate publication; to insure publication the last discussion should be submitted by August, 1942.

[1] Research Engr. and Associate Prof., Math. and Civ. Eng., Colorado State College, Fort Collins, Colo.

differential equation of gradually varied flow, the writer found certain irregularities in the results. These irregularities consisted of deviations from the general shape for such curves as found in the standard texts treating the subject of gradually varied flow. The deviations occurred in the case where the flow was at depths less than either the critical depth y_c or the normal depth y_o. (Hereafter, in this paper, curves satisfying these conditions of flow will be indicated by the subscript 3;[2,3] that is, in the case of a mild slope, $y_c < y_o$, that part of the profile of the surface of flow for depth less than y_c will be indicated by the symbol M_3.) A point of inflection appears in the calculated curves of this type (3), whereas such a point is not shown in the usual textbook graphs. Although not so stated in most of these texts, perhaps the surface profiles for gradually varied flow have been sketched from a combination of certain qualitative features of the equation, plus a knowledge of the forms which these curves should take. In this paper the usual differential equation will be re-derived, giving the result in a lucid form, and the correct profile curves corresponding to this equation will be drawn, assuming a Chézy coefficient varying according to the Manning, Bazin, or Ganguillet-Kutter relations. Further, the related academic problem of integrating the differential equation of gradually varied flow with a variable Chézy coefficient as given by the Manning relation will be solved for the case of a wide rectangular channel.

It is recalled that the general equation of varied flow is obtained as follows:[2] The mean total head E_w at a cross section is given by,

$$E_w = \frac{V^2}{2\,g} + y + h_o \qquad (1)$$

in which V is the velocity of flow, g is the gravitation constant, y is the depth of flow, and h_o is the height of the bottom of the channel above an arbitrary horizontal datum. Differentiation of this equation with respect to x, the distance measured in the direction of flow, gives:

$$\frac{dE_w}{dx} = \frac{d}{dx}\left(\frac{V^2}{2\,g}\right) + \frac{dy}{dx} + \frac{dh_o}{dx} \qquad (2)$$

In this equation the expression $\frac{dE_w}{dx}$, representing the rate of change of mean total head, is assumed to be the same as if the flow were uniform, and hence, by the Chézy relation, will be equal to $-\frac{V^2}{C^2\,R}$, C being the Chézy coefficient and R the hydraulic radius for the section of the channel under discussion. If b is the breadth, A the cross-section area, and Q the total discharge across the section of the channel, then the expression $\frac{d}{dx}\left(\frac{V^2}{2\,g}\right)$ can be modified by

[2] See, for example, "Fluid Mechanics for Hydraulic Engineers" by Hunter Rouse, McGraw-Hill Book Co., Inc., New York, N. Y., and London, England, 1938, 1st Ed., pp. 288 *et seq.*

[3] "Hydraulics of Open Channels," by Boris A. Bakhmeteff, McGraw-Hill Book Co., Inc., New York, N. Y., 1932 (see types 1 and 2).

making $V = \frac{Q}{A}$ and noting that $dA = b\,dy$; whence,

$$\frac{d}{dx}\left(\frac{V^2}{2\,g}\right) = -\frac{Q^2\,b}{g\,A^3}\frac{dy}{dx} \qquad (3)$$

If S_o is the slope of the channel bottom (positive for a downward slope in the direction of flow), then $\frac{dh_o}{dx} = -S_o$, and Eq. 2 becomes,

$$-\frac{V^2}{C^2\,R} = -\frac{Q^2\,b}{A^3\,g}\frac{dy}{dx} + \frac{dy}{dx} - S_o \qquad (4)$$

or

$$\frac{dy}{dx} = \frac{S_o - \frac{V^2}{C^2\,R}}{1 - \frac{Q^2\,b}{A^3\,g}} \qquad (5)$$

Since in the remainder of this paper the writer is interested only in the case of the broad rectangular channel, the general Eq. 5 will now be simplified accordingly. It follows at once that for such a channel $R = y$, and that for a unit width of the channel $A = y$ (Q will be retained but is understood now to be the discharge per unit width of channel).

Remembering that the height H of the energy line above the channel bottom is given by $H = y + \frac{V^2}{2\,g}$, that $V = \frac{Q}{y}$, and that $\frac{dH}{dy}$ is equal to zero at the conditions of critical flow:

$$\frac{dH}{dy} = \frac{d}{dy}\left(y + \frac{Q^2}{2\,g\,y^2}\right) = 1 - \frac{Q^2}{g\,y^3} \qquad (6)$$

Hence, at critical conditions of flow,

$$y^3_c = \frac{Q^2}{g} \qquad (7)$$

y_c being the critical depth. This reduces the denominator of the right member of Eq. 5 to $1 - \frac{y^3_c}{y^3}$.

In a similar manner, by noting that at the normal depth y_o the energy loss $-\frac{dE_w}{dx} = -S_o$, it is seen that, for normal conditions of flow,

$$S_o = \frac{V^2}{C^2_o\,y_o} = \frac{Q^2}{C^2_o\,y^3_o} \qquad (8)$$

or

$$y^3_o = \frac{Q^2}{S_o\,C^2_o} \qquad (9)$$

in which C_o is the value of C when flow is at normal depth. Whence the numerator of Eq. 5 becomes:

$$S_o - \frac{V^2}{C^2 y} = S_o - \frac{Q^2}{C^2 y^3} = S_o - \frac{S_o C^2_o y^3_o}{C^2 y^3} = S_o \left(1 - \frac{C^2_o y^3_o}{C^2 y^3}\right) \quad (10)$$

and Eq. 5 becomes:

$$\frac{dy}{dx} = \frac{S_o \left(1 - \frac{C^2_o y^3_o}{C^2 y^3}\right)}{1 - \frac{y^3_c}{y^3}} \quad (11a)$$

In Eq. 11*a* the relation of the slope of the surface profile to the critical and normal depths appears in easily recognized form. This equation is valid when S_o is positive, negative, or zero. In the latter case, the equation simplifies to:

$$\frac{dy}{dx} = \frac{- g\, y^3_c}{C^2 (y^3 - y^3_c)} \quad (11b)$$

The factor C may now be replaced by any one of the three expressions for it as given by Manning, Bazin, or Ganguillet-Kutter. For the moment the Manning relation

$$C = 1.486 \frac{R^{1/6}}{n} \quad (12)$$

will be used. This value, when substituted in Eq. 11*a*, gives at once,

$$\frac{dy}{dx} = S_o \left[\frac{1 - \left(\frac{y_o}{y}\right)^{10/3}}{1 - \left(\frac{y_c}{y}\right)^3}\right] \quad (13)$$

From Eq. 13 it is seen at once that the surface is vertical (a theoretical condition only, since the postulated uniform flow condition is not satisfied here) not only at the critical value but also at $y = 0$. Consequently, there must be a point of inflection in the type 3 curve (except when y_c is greater than, or equal to, y_o). To find the location of this point of inflection, Eq. 13 is differentiated with respect to x and the result is equated to zero. This gives:

$$9\, y^3_c\, y^{10/3} - 10\, y^{10/3}_o \; y^3 + y^{10/3}_o \; y^3_c = 0 \quad (14)$$

The solution of Eq. 14 for y is seen to depend upon both y_o and y_c. If y_c is much less than y_o, the resulting value of y is small and this point of inflection is near the bottom of the channel; but, if y_c approaches y_o, the value of y also approaches y_o. Even for the value $y_c = \frac{y_o}{2}$, the inflection occurs at about $y = \frac{y_o}{4}$.

However, if the value of C, as given by either the Bazin or Ganguillet-Kutter relations, is used in Eq. 11a, the inflection point, although still existing, is so close to the bottom of the channel that any error introduced by ignoring it is practically negligible. For the case of $y_c = \frac{y_o}{2}$, the point of inflection occurs at a value of y that is less than 1% of y_o. From the foregoing discussion two conclusions can be drawn at once:

(1) If step-by-step integration is used in the range of type 3 curves, the Chézy coefficient C should be determined by either the Bazin or Ganguillet-Kutter relation and not by the Manning relation; and

(2) Perhaps the backwater curves as drawn in the standard texts are based upon a constant Chézy coefficient C rather than upon one given by the usual Manning, Bazin, or Ganguillet-Kutter relations.

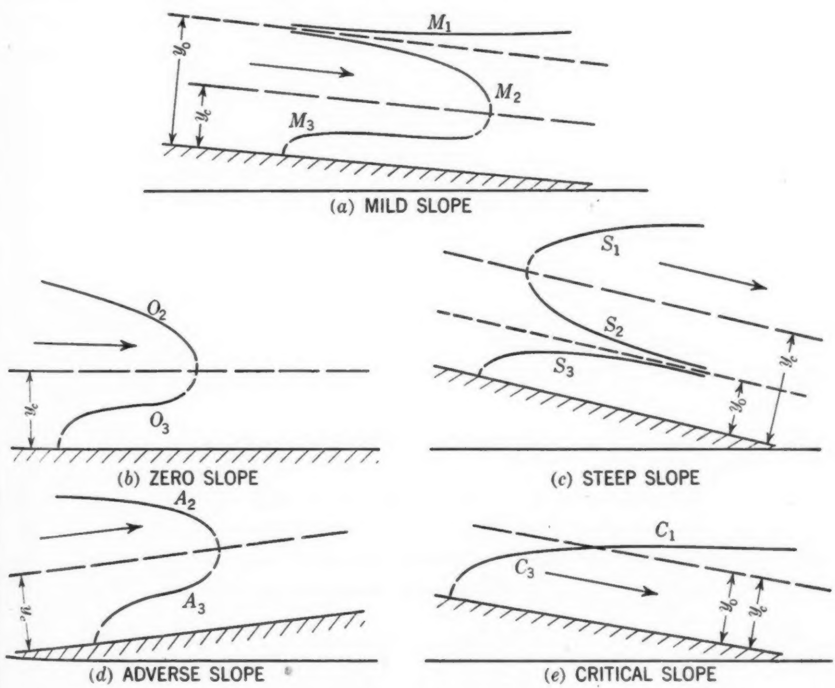


FIG. 1.—SURFACE PROFILES FOR GRADUALLY VARIED FLOW $C = 1.486 \frac{R^{1/6}}{n}$

In order to illustrate the contrast between the actual forms of the surface profiles for gradually varied flow in a broad rectangular channel, using a variable Chézy coefficient as determined by the Manning relation, and the forms of these profiles as sketched in texts for a constant Chézy coefficient, the sketches of the former curves are presented in Fig. 1 to be contrasted with the latter curves as found in textbooks.

The differential Eq. 13 is not difficult to integrate. In fact it may be rewritten in the form:

$$S_o \, dx = \frac{y^{1/3} (y^3 - y^3{}_c) \, dy}{y^{10/3} - y^{10/3}{}_o} \quad \text{(15)}$$

or

$$S_o (x + C_1) = y + \int \frac{- y^{1/3} \, y^3{}_c + y^{10/3}{}_o}{y^{10/3} - y^{10/3}{}_o} \, dy$$

$$= y + \frac{1}{2 \, y^{5/3}{}_o} \int \frac{- y^{1/3} \, y^3{}_c + y^{10/3}{}_o}{y^{5/3} - y^{5/3}{}_o} \, dy - \frac{1}{2 \, y^{5/3}{}_o} \int \frac{- y^{1/3} \, y^3{}_c + y^{10/3}{}_o}{y^{5/3} - y^{5/3}{}_o} \, dy \quad \text{(16)}$$

Although the details are somewhat tedious, each of these integrals can be evaluated by partial fractions. The result is:

$$x = \frac{y}{S_o} + 0.306 \, \frac{y^3{}_c}{S_o \, y^2{}_o} + \frac{0.3}{S_o \, y^2{}_o} \, K \quad \text{(17)}$$

in which K has the following value:

$$\begin{aligned}
K = {} & \\
& (0.809 \, y^3{}_c - 0.309 \, y^3{}_o) \log_e (y^{2/3} - 1.618 \, y^{1/3} \, y^{1/3}{}_o + y^{2/3}{}_o) \\
& + (0.809 \, y^3{}_c + 0.309 \, y^3{}_o) \log_e (y^{2/3} + 1.618 \, y^{1/3} \, y^{1/3}{}_o + y^{2/3}{}_o) \\
& - (0.309 \, y^3{}_c + 0.809 \, y^3{}_o) \log_e (y^{2/3} - 0.618 \, y^{1/3} \, y^{1/3}{}_o + y^{2/3}{}_o) \\
& - (0.309 \, y^3{}_c - 0.809 \, y^3{}_o) \log_e (y^{2/3} + 0.618 \, y^{1/3} \, y^{1/3}{}_o + y^{2/3}{}_o) \\
& - (y^3{}_c + y^3{}_o) \log_e |y^{1/3} + y^{1/3}{}_o| - (y^3{}_c - y^3{}_o) \log_e |y^{1/3} - y^{1/3}{}_o| \\
& + (1.176 \, y^3{}_c - 1.902 \, y^3{}_o) \tan^{-1} (y^{1/3} - 0.809 \, y^{1/3}{}_o) \frac{1}{0.588 \, y^{1/3}{}_o} \\
& - (1.176 \, y^3{}_c + 1.902 \, y^3{}_o) \tan^{-1} (y^{1/3} + 0.809 \, y^{1/3}{}_o) \frac{1}{0.588 \, y^{1/3}{}_o} \\
& - (1.902 \, y^3{}_c - 1.176 \, y^3{}_o) \tan^{-1} (y^{1/3} - 0.309 \, y^{1/3}{}_o) \frac{1}{0.951 \, y^{1/3}{}_o} \\
& + (1.902 \, y^3{}_c + 1.176 \, y^3{}_o) \tan^{-1} (y^{1/3} + 0.309 \, y^{1/3}{}_o) \frac{1}{0.951 \, y^{1/3}{}_o} \quad \text{(18)}
\end{aligned}$$

In the formula for the substitution factor K (see Eq. 18), the various constants appearing are approximations for the following:

Constant	Quantity
0.306	$\frac{3\pi}{25}$ (3 sin 36° − sin 72°)
0.809	cos 36°
0.309	cos 72°
1.618	2 cos 36°
0.618	2 cos 72°
1.176	2 sin 36°
1.902	2 sin 72°
0.588	sin 36°
0.951	sin 72°

All these constants arise from the integration and the determination of the constant of integration so that the origin will be at the intersection of the type 3 curve with the bottom of the channel. It is readily seen that several of the terms appearing in Eq. 18 could be combined; but the given form lends itself somewhat more readily to calculation. Furthermore, the logarithms have been left to the base e to facilitate work with the slide rule, by which satisfactory results can be obtained quite quickly.

The foregoing solution enables one to draw quantitative graphs of the various profile curves for the channel type considered here. Further, although it is not practical for accurate calculations of flow in natural stream beds, the equation is convenient for making rough approximations in such cases and for use in certain artificial stream beds. It is to be noted, of course, that this result, depending upon the Manning relation for C, is subject to the same restriction in the region of the type 3 curve as was mentioned previously herein.

The integration for zero and negative channel slopes is included in the integrated forms 17 and 18. However, the indeterminate forms appearing for the case of zero slope make it advisable to integrate Eq. 11*b* for this case. This integration is elementary and leads to the profile equation,

$$7.87\, g\, n^2\, y^3_c\, x = 13\, y^{4/3}\, y^3_c - 4\, y^{13/3} \qquad (19)$$

in which n is the Manning roughness factor.

APPENDIX

NOTATION

The following letter symbols conform essentially with the Society's *Manual of Engineering Practice No. 11*, on "Letter Symbols and Glossary for Hydraulics," and American Standard Letter Symbols for Hydraulics, prepared by a Committee of the American Standards Association, with Society representation, and approved by the Association in 1942:[4]

A = the area of the cross section of the flow;
b = the breadth of the channel;
C = the Chézy coefficient: C_o = the Chézy coefficient for normal depth of flow;
E_w = mean total head;
g = the gravitation constant;
H = the height of the specific energy line above the channel bottom;
h_o = the height of the channel bottom above an arbitrary horizontal datum;

[4] ASA—Z10.2—1942.

K = a simplifying, substitution factor (Eqs. 17 and 18);
M_3 = that part of the surface profile for a mild slope and depth of flow less than the critical depth;
n = Manning's roughness factor;
Q = total quantity of flow:
R = the hydraulic radius;
S_o = the slope of the channel bottom with respect to the horizontal;
V = the velocity of flow;
x = the distance measured parallel to the channel bottom;
y = the depth of the flow at any section:
y_c = the depth of flow for critical flow conditions; and
y_o = the normal depth of flow.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

PAPERS

MODEL TESTS ON STRUCTURES FOR HYDROELECTRIC DEVELOPMENTS

By L. M. Davis,[1] Assoc. M. Am. Soc. C. E.

Synopsis

Two series of model tests on hydraulic structures are described in this paper. These are the Holtwood and Safe Harbor hydroelectric developments on the Susquehanna River in southern Pennsylvania near the Maryland border. Data are presented on tests made to determine: (*a*) The discharge coefficients of various spillways; (*b*) the effect of depth of approach on the discharge coefficient; (*c*) the effect of submerged obstructions, such as cofferdam cribs, on discharge; (*d*) the water pressures acting on the spillway face for various conditions; (*e*) the effectiveness of various pier-nose designs; (*f*) the results of erosion in connection with apron design; and (*g*) the proper location of a deflecting wall below the Holtwood Dam.

Introduction

Hydraulics has been the subject of experimentation for many centuries. During the past century human knowledge concerning the science of hydraulics has increased at a rapid rate, largely due to work done in laboratories throughout the world. Nevertheless, many problems confront the hydroelectric designer, for which a satisfactory and economical solution can be reached only by means of laboratory experiments. Few hydroelectric developments have been built in recent years that were not preceded by a comprehensive testing program.

A number of problems arose in connection with the design of the hydraulic structures for the Safe Harbor development that required model tests in order to insure a satisfactory and economical structure. Consequently, a test program was conducted during the summer of 1930 that included tests on spillway sections, numerous designs of pier noses, and tests to determine a suitable apron to be installed below the dam to prevent the flood waters from causing erosion which would eventually endanger the dam.

Note.—Written comments are invited for immediate publication; to insure publication the last discussion should be submitted by **August, 1942.**

[1] Hydr. Test Engr., Pennsylvania Water & Power Co., Holtwood, Pa.

Necessity for model tests on the Holtwood development arose in 1936. The spring flood of that year on the Susquehanna River reached a peak of 860,000 cu ft per sec, which was 20% greater than the value used for the design of the Holtwood development. Following this flood it was decided to make alterations to the hydraulic structures at Holtwood so that they would withstand safely a flood of 1,200,000 cu ft per sec, the value which was used in the design of the Safe Harbor development. It was recognized that under this flow certain parts of the Holtwood Dam would be exposed to negative pressures. To determine the extent and magnitude of these negative pressures, it was decided to conduct model tests. It was also desired to obtain, by means of model tests, a satisfactory and economical design of deflection wall to separate the main river channel from the tailrace and to replace the wall which failed during the flood. An apron had never been provided below the Holtwood Dam. On three occasions since the dam was completed in 1910, paving has been necessary in several areas along the toe of the dam to repair the effects of erosion. It was decided to design a suitable apron that could be installed if repairs should become necessary in the future.

Description of Models

All spillway calibrations made in 1930 for the Safe Harbor series of tests were done in a 3-ft by 35-ft glass-sided wooden flume (see Fig. 1) in the basement of the main laboratory building. The model installed in this flume was a 1 : 40 scale ratio that provided one full spillway opening and two half spillway openings.

For the erosion tests a 1 : 40 ratio model was installed in the main creek below the laboratory as shown in Figs. 2 and 3. This model was provided with five full spillway openings and two half openings so that the effect of various combinations of "gates open" could be determined. The erosion bed was composed of 1¼-in. to 2-in. gravel.

For the 1936 series of tests for the Holtwood development, it was decided to take advantage of a 5-ft concrete flume, located parallel to the creek just below the main laboratory, for the 1 : 30 scale model that was used for calibration, measurement of pressures on the surface of the dam, and for erosion tests. The setup is shown in Fig. 4. There was sufficient room between this 5-ft flume and the creek for the 1 : 50 scale model that was used for the deflection-wall tests. This was a true model of 300 ft of the east end of the dam, east abutment, and deflection wall below the dam. Water for both of these model setups was obtained through a 30-in. pipe from the main laboratory building. This pipe started at a point next to the 50,000-lb weighing tank so that all flows could be checked accurately.

Coefficients of Discharge for Various Spillway Sections Tested

In connection with the tests for the Safe Harbor Dam the coefficients of discharge

$$C = \frac{Q}{L\,H^{1.5}} \qquad (1)$$

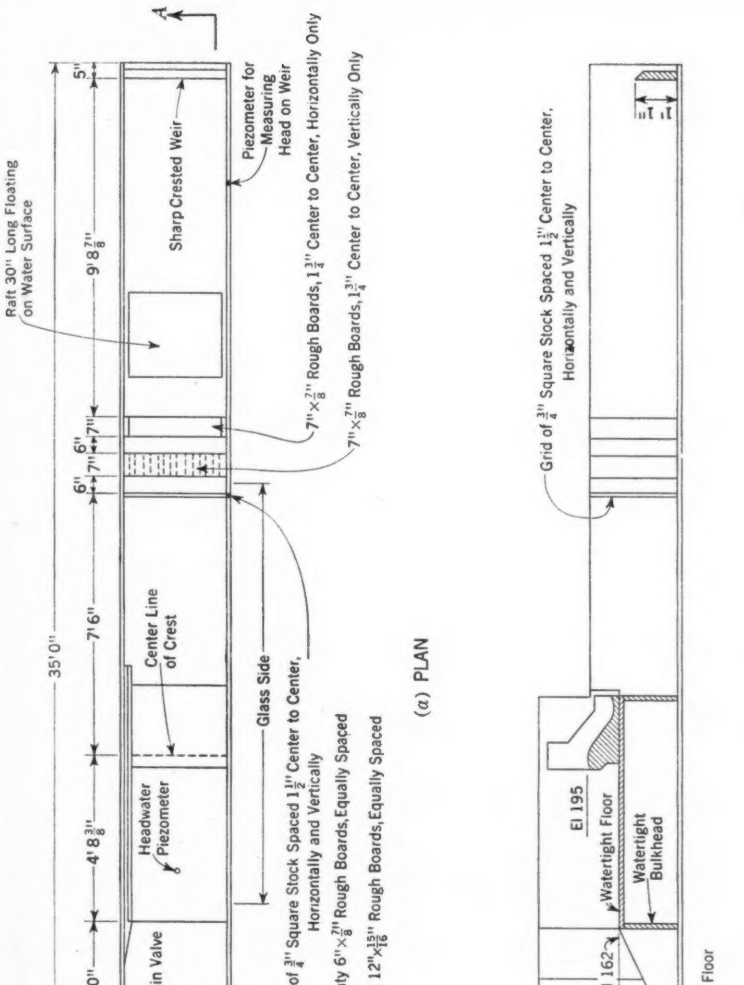


FIG. 1.—MODEL SETUP FOR CALIBRATION AND PIER-NOSE TESTS, SAFE HARBOR, PA.

of four proposed spillway sections were determined. The results of these tests are shown in Fig. 5. In Eq. 1, L, the width of one full gate and two half gate openings, equals 2.392 ft; H equals one fortieth of the prototype head; and the approach area, in square feet, equals 2.819 (H + 0.838). In first studying these results it did not seem logical that, at designed head, the shorter dam nose (curve 3, Fig. 5) proposed by I. A. Winter, Assoc. M. Am. Soc. C. E., should give a higher coefficient of discharge than Design 2,[2] which follows the

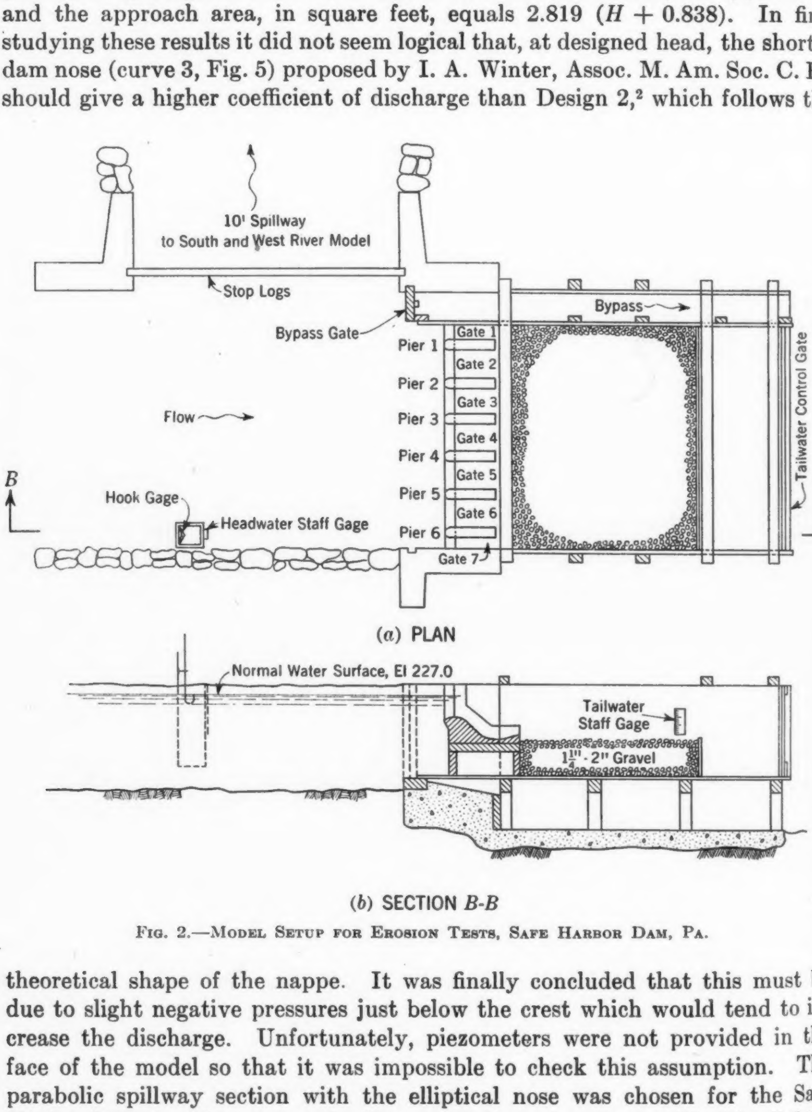


FIG. 2.—MODEL SETUP FOR EROSION TESTS, SAFE HARBOR DAM, PA.

theoretical shape of the nappe. It was finally concluded that this must be due to slight negative pressures just below the crest which would tend to increase the discharge. Unfortunately, piezometers were not provided in the face of the model so that it was impossible to check this assumption. The parabolic spillway section with the elliptical nose was chosen for the Safe Harbor Dam to effect economies in construction, since even with this slightly less efficient design the proposed thirty-two gates would discharge somewhat more than the maximum estimated river flow.

[2] "Hydraulic Handbook," by William P. Creager and Joel D. Justin, 1927 Ed., John Wiley & Sons, Inc., New York, N. Y., Table 32, p. 209.

Experience at the Holtwood Dam indicated that a certain degree of silting would occur in the pool above the dam. With this in mind a series of tests was made to determine the effect on the discharge coefficients of various elevations of the approach channel. The coefficients obtained in these tests are shown in Fig. 6, both with and without correction for the velocity of approach. Corrected for velocity of approach, the coefficient is defined by

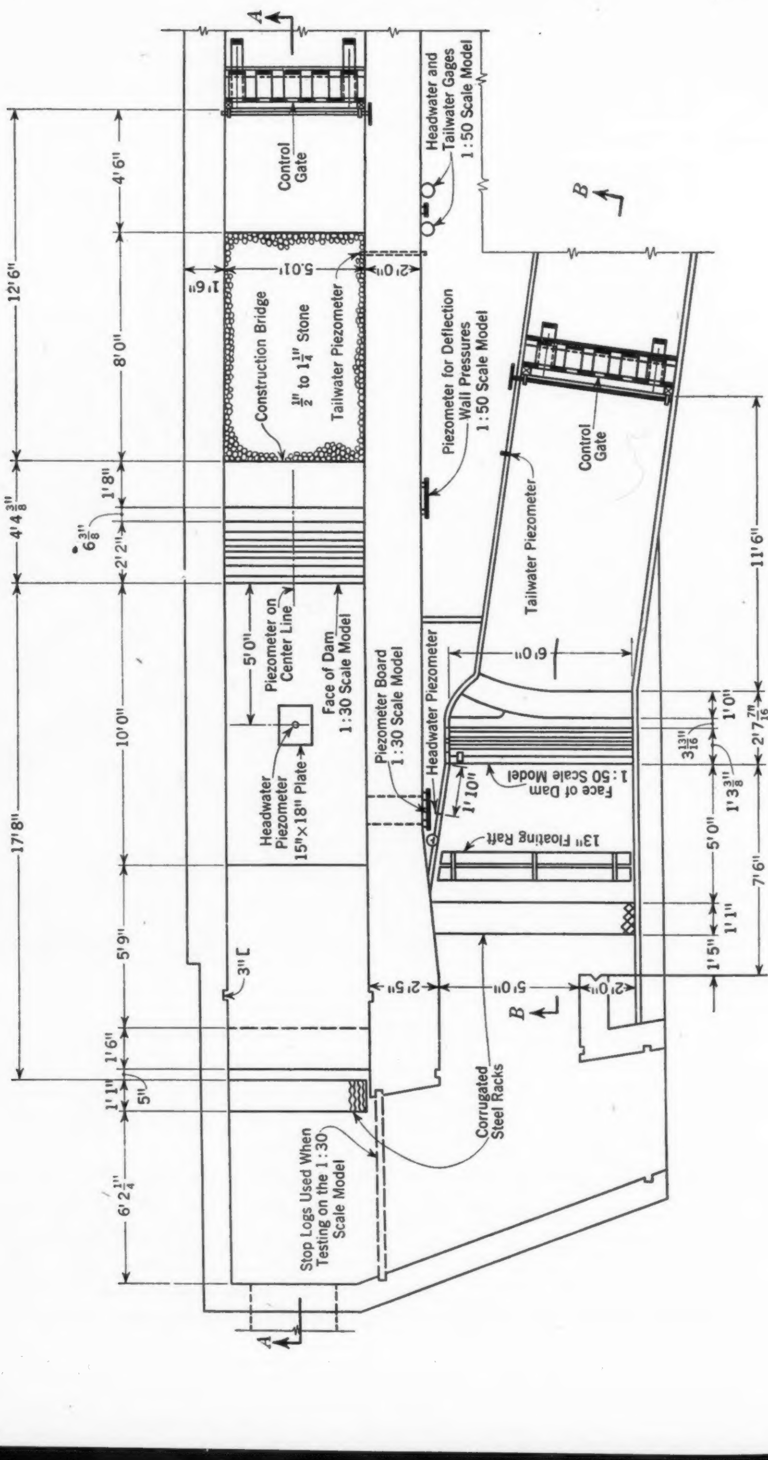
$$C = \frac{Q}{L\left[\left(H + \frac{V^2}{2g}\right)^{1.5} - \frac{V^2}{2g}\right]} \qquad (2)$$

Without correction, the coefficient is defined by Eq. 1. In computing the dis-

FIG. 3.—CHARACTERISTIC FLOW DURING EROSION TESTS, SAFE HARBOR DAM, PA.

charge curve for various gates open at Safe Harbor, a partial correction for velocity of approach was made, since with all gates open the width of spillway is only about 50% of the total river width. An additional correction was found necessary to take care of the excessive end contraction caused by crossflow when a large number of adjacent gates are open.

The necessity for this correction can be realized from an inspection of Fig. 7, which was taken from the hill above the west abutment of the Safe Harbor Dam. The river flow was 486,000 cu ft per sec, and twelve floodgates were open in the west spillway. In the foreground on the upstream side of the dam, it will be noted that there is a heavy flow indicated along the face of the bulkhead section toward the first gate. This crossflow is sufficient to cause a



(a) PLAN OF MODELS

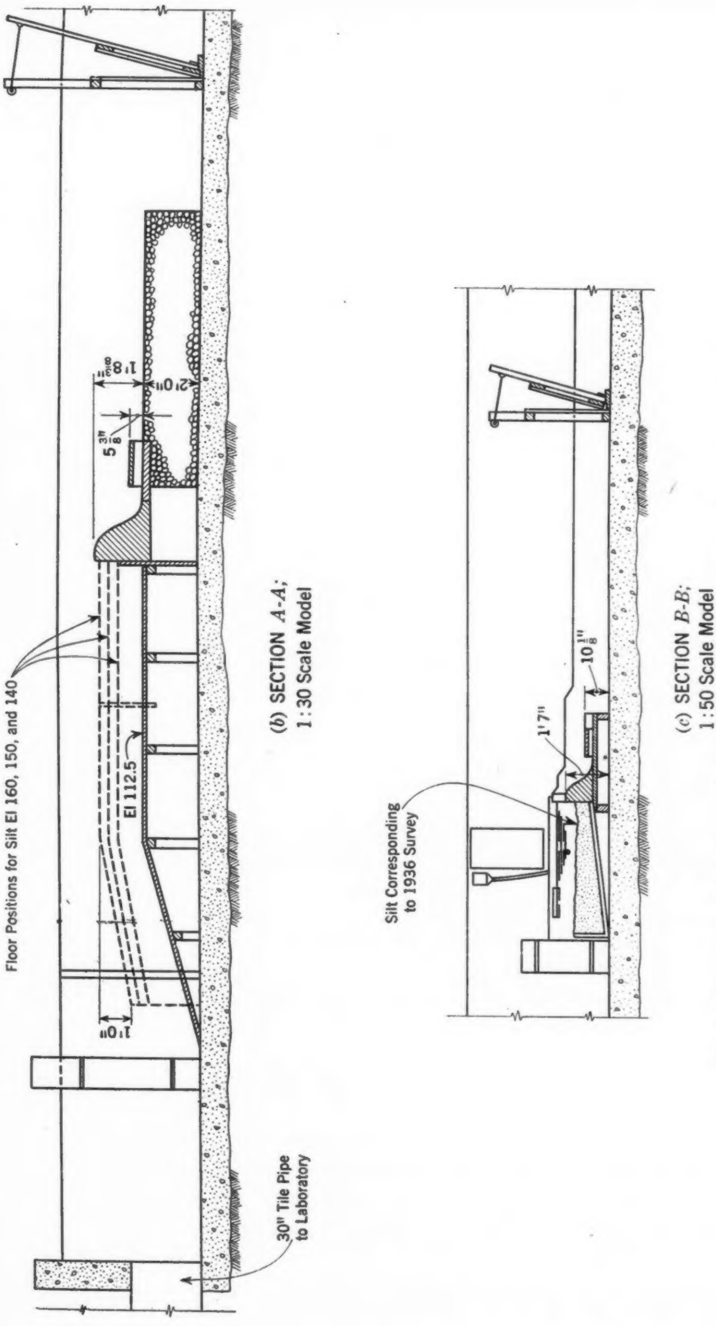


FIG. 4.—LAYOUT OF MODELS FOR SPILLWAY AND DEFLECTION-WALL TESTS, HOLTWOOD DAM, PA.

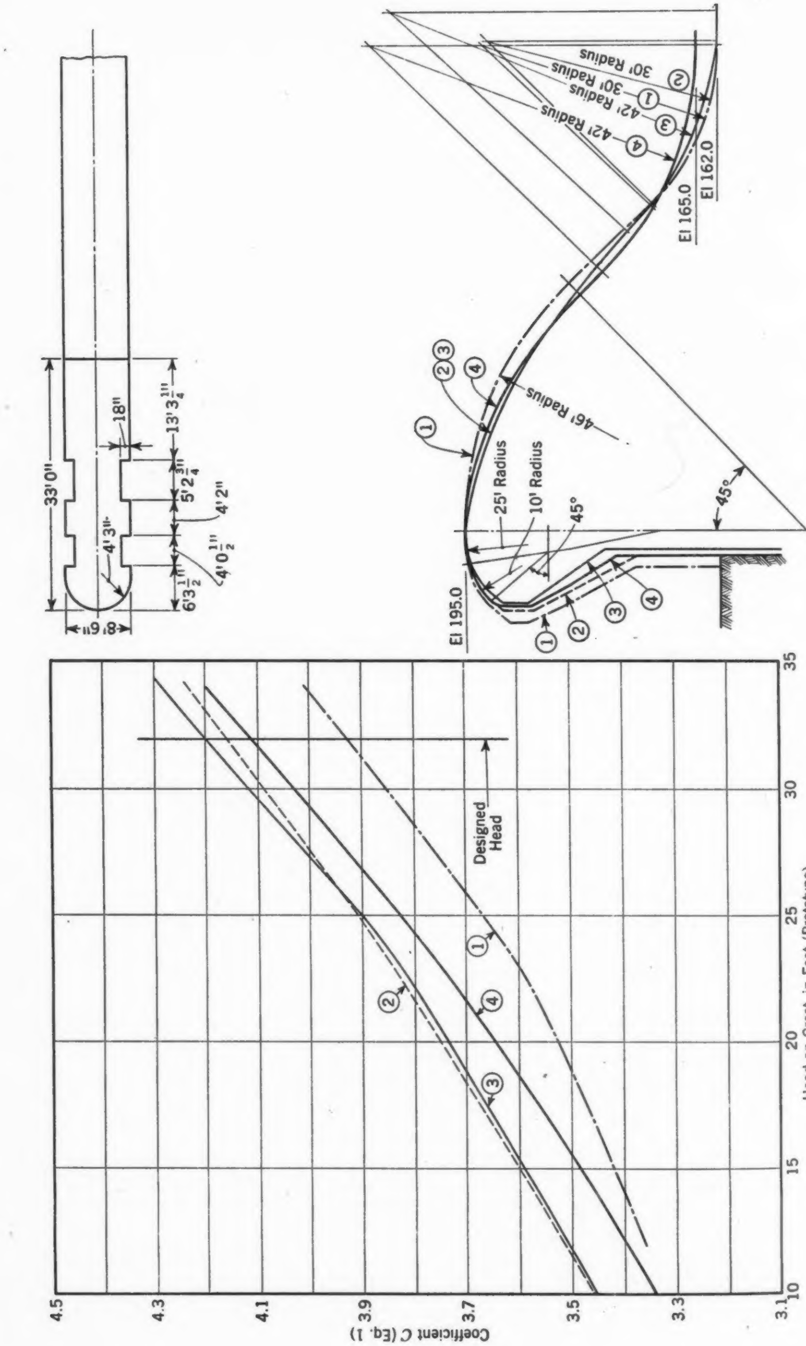


FIG. 5.—COMPARISON OF DISCHARGE COEFFICIENTS FOR VARIOUS SPILLWAY SECTIONS

considerable drawdown on the near side of the first gate and some piling up of the water on the far side. The water is deflected from the pier on the far side of this opening, so that by the time it reaches the bottom of the bucket it has been thrown to the near side of the opening and forms a huge wave where it hits the pier extension. Judging from the wave pattern of the water leaving the apron, the crossflow has some slight effect even on the fourth gate from the ends. It is evident that this crossflow must reduce the discharge of the end-gates appreciably when a considerable number of adjacent gates are open.

At first glance it is difficult to understand why the coefficients of discharge with the velocity-of-approach correction improve with increase in velocity of approach in the case of the Safe Harbor Dam (Fig. 6). The only reasonable explanation seems to be that the recessed face must cause unfavorable flow conditions that increase the losses with increase in depth of approach. It is known that the discharge over a sharp-crested weir is reduced when the face of the weir is inclined upstream.

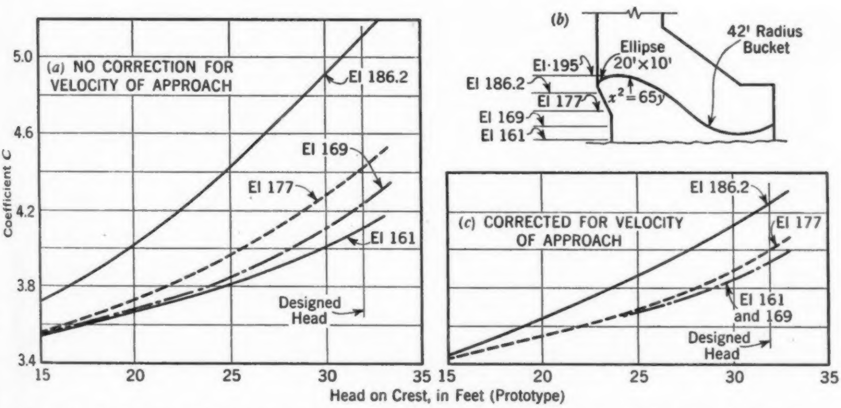


FIG. 6.—DISCHARGE COEFFICIENTS FOR VARIOUS SILT ELEVATIONS IN THE APPROACH CHANNEL, SAFE HARBOR DAM

The results of the calibration tests made on the Holtwood model dam, with various elevations of silt in the approach channel, are shown in Fig. 8. The coefficients of discharge are given with no correction (Eq. 1) and also with full correction for velocity of approach (Eq. 2). In computing the discharge curve for the Holtwood Dam, no correction for velocity of approach was applied, since there are no piers or floodgates on the dam and the approach channel is approximately the same width as the overflow section.

In arriving at the discharge curves for the Safe Harbor and Holtwood spillways, the coefficients of discharge as obtained from the model tests were used without any "step-up." (The term "step-up" is in common use in hydraulic turbine work. Experience with hydraulic turbines has shown that in a homologous series the percentage of losses varies inversely with the size. Since model tests are made on a small size turbine the term "step-up" is applied to the correction necessary to predict the characteristics of the prototype accurately.)

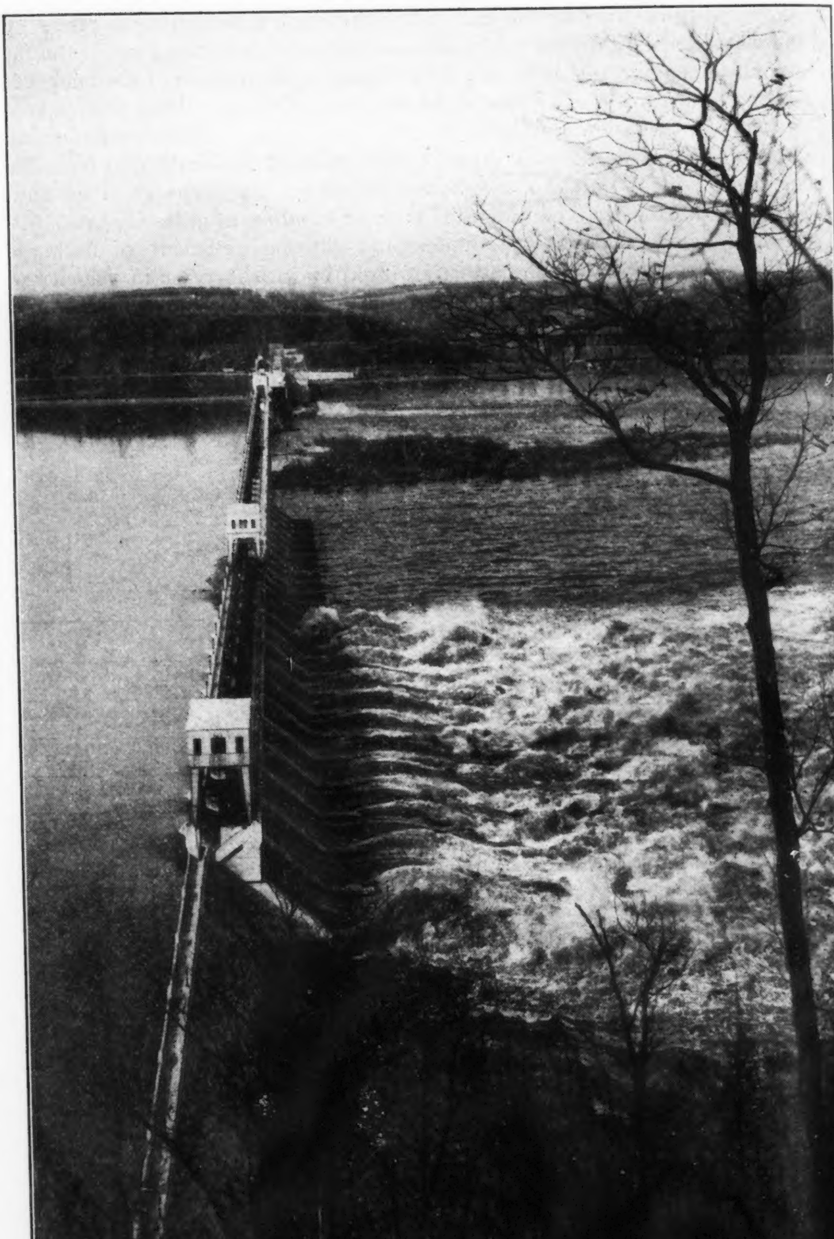


FIG. 7.—VIEW OF SAFE HARBOR DAM FROM THE WEST ABUTMENT

It seems probable that, if the same ideal flow conditions existed on the prototype that existed on the model, some step-up would be obtained. However, ideal flow conditions do not usually exist at flood stages so that it would seem unwise to claim any step-up in preparing prototype discharge curves.

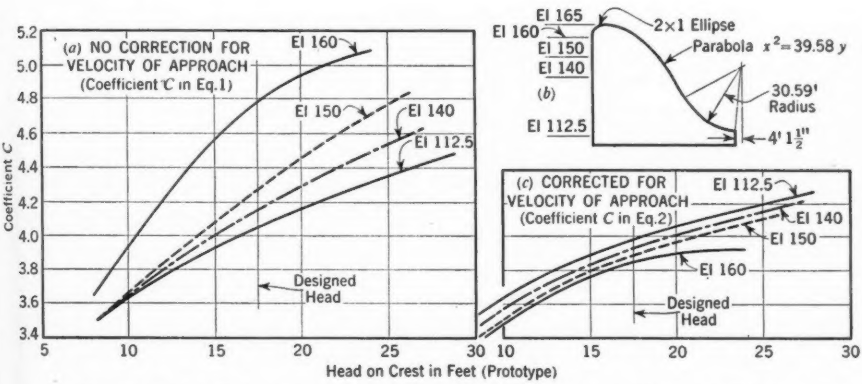


FIG. 8.—DISCHARGE COEFFICIENTS FOR VARIOUS SILT ELEVATIONS IN THE APPROACH CHANNEL, HOLTWOOD DAM

During the 1936 flood there was an excellent opportunity to obtain a comparison between the Susquehanna River discharge as obtained at the Safe Harbor, the Holtwood, and the Conowingo, Md. (14 miles downstream from Holtwood), hydro plants. The flood peak measured at Holtwood was about 1.5% less than the Conowingo measurement and 3.0% less than the Safe Harbor measurement. The Safe Harbor measurement was based on curves that did not take into consideration the excessive end contraction shown in Fig. 7, so it is not surprising that Safe Harbor indicated more water than the other two plants. It is felt that this comparison shows remarkable agreement, considering that the flow was 860,000 cu ft per sec.

EFFECT OF COFFERDAM CRIBS ON SPILLWAY DISCHARGE

Following the completion of the Safe Harbor Dam, it was found that it would be a rather expensive operation to remove the cribs of the upstream cofferdam entirely. To determine the effect that these cribs might have on spillway discharge, a series of tests was made, the results of which are shown in Fig. 9. With the river bottom at El. 165, tests were made with the cribs and stop logs in place, with the cribs in place and the stop logs removed, and with a solid cofferdam, making tests for each case at various stages of removal. Tests were repeated with a river bottom at El. 175 for the cribs without stop logs. It will be noted that the loss in discharge with the cribs and stop logs in place to El. 193 is 18%, whereas the loss with the cribs in place to El. 193 with stop logs removed is only 4.1%. Since it was relatively easy, the stop logs were completely removed from the cofferdam and the cribs were removed to El. 184. This did not involve any underwater work, and the loss in discharge

for the cribs in place to this elevation is only 1.7%. Considerable silt has been deposited in the pond but apparently the cribs have set up a disturbance that has kept the river bed clear between the cribs and the dam.

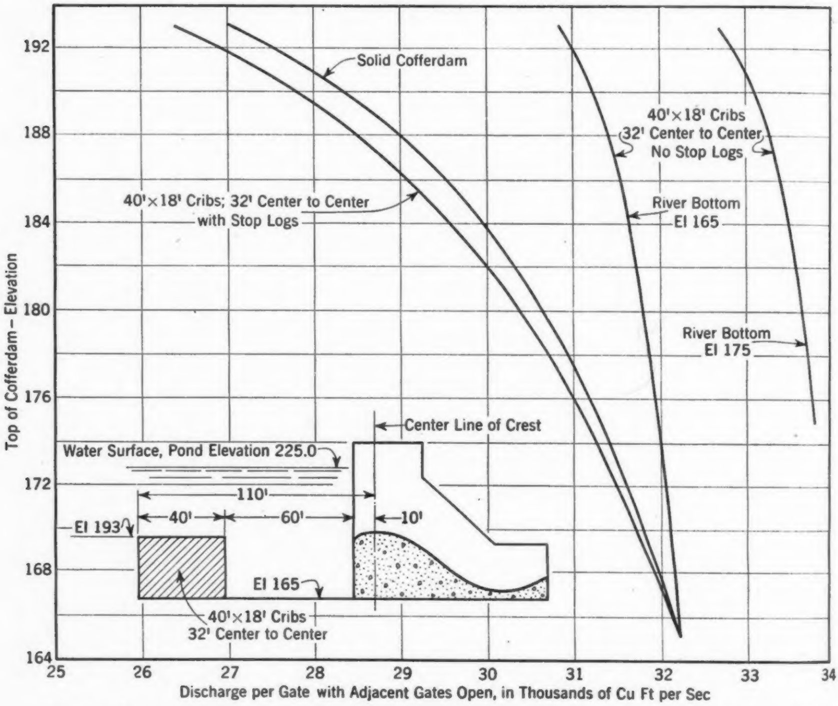


Fig. 9.—Effect on Discharge by Cofferdam Cribs Left in Place to Various Elevations, Safe Harbor Dam

Pressure Measurement on the Face of the Dam

Before accepting the parabolic section for the face of the Safe Harbor Dam, it was deemed advisable to determine the water pressures acting on the face of the dam under designed head condition. From an inspection of Fig. 5 it will be noted that the slope of the parabolic face as it approaches the bucket is somewhat greater than the slope of the theoretical lower nappe. There was some concern that this might result in negative pressures; but the results of the measurement of water pressures on the model (see Figs. 10 and 11) show that this fear was unfounded. It will be noted that the back pressure, resulting from the bucket deflecting the jet of water, extends some distance upstream from the beginning of the bucket. No doubt this back pressure, combined with the friction loss between the water and the face of the dam, is responsible for the rather high positive pressures at a point where negative pressures might be expected.

One spillway section of the Safe Harbor Dam was provided with five piezometers along the center line. It will be noted that the pressures obtained from

the prototype agree reasonably well with the pressures obtained on the model at and near the crest of the dam, but diverge widely as the bucket is approached. Of course, part of this divergence is due to the fact that the elevation at the

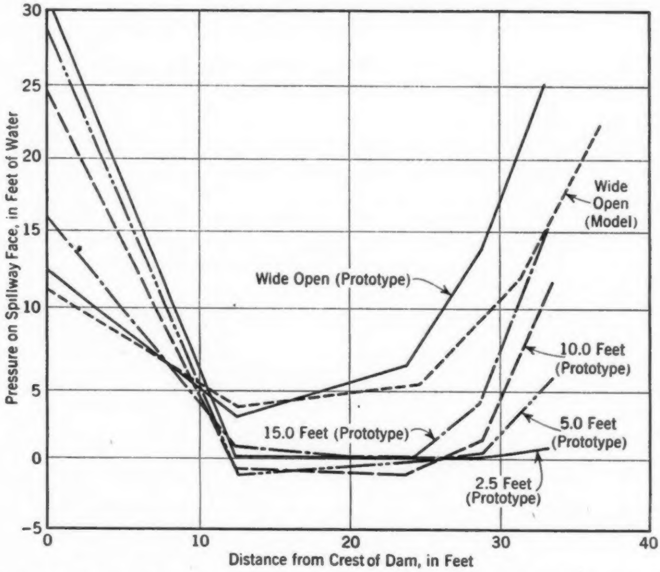


FIG. 10.—PRESSURE ON THE FACE OF THE SPILLWAY, SAFE HARBOR DAM, FOR VARIOUS GATE POSITIONS

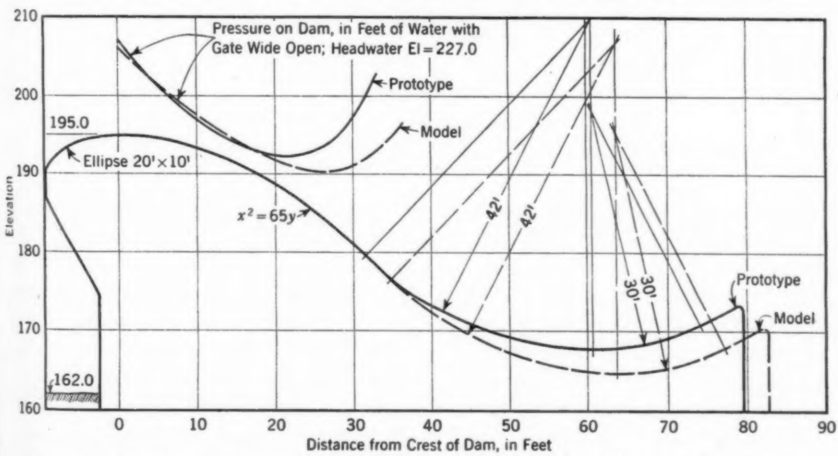


FIG. 11.—COMPARISON OF WATER PRESSURES ON THE SPILLWAY FACE OF MODEL AND PROTOTYPE, SAFE HARBOR DAM

bottom of the bucket of the model was 165, whereas the elevation of the bottom of the bucket of the prototype was 168. The elevation of the water below the dam for the prototype tests was about 5 ft higher than for the model tests, which also might account for some of this discrepancy.

When the model tests were conducted for Safe Harbor, it was felt that none of the standard floodgates would be operated at part gate opening, since four double-leaf regulating gates, which can be operated by remote control from the operating room, were provided adjacent to the power house for pond-level control. After Safe Harbor went into service, it was discovered that a slight gain in head could be obtained by doing some of the regulating with one of the standard floodgates at the west end of the dam. Consequently pressure measurements were made on the prototype for various part gate positions. From Figs. 10 and 11 it will be noted that slight negative pressures existed over part of the spillway face. However, these may be considered of little concern since the dam face is vented through end contraction of the jet.

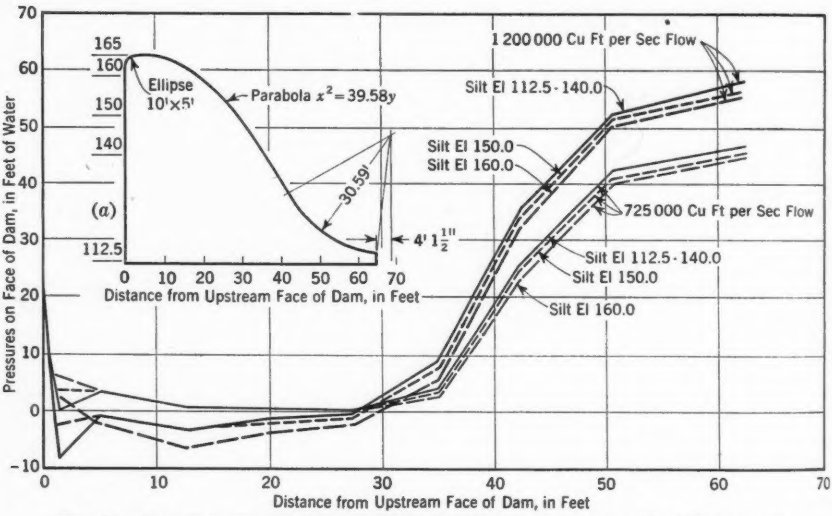


FIG. 12.—PRESSURE ON THE FACE OF HOLTWOOD DAM WITH VARIOUS SILT ELEVATIONS IN THE APPROACH CHANNEL

The results of the pressure measurements on the face of the Holtwood spillway model are shown in Fig. 12. For these tests the effect of depth of the approach channel on the pressures was investigated. The Holtwood Dam was designed for a peak river flow of 725,000 cu ft per sec. It will be noted that for this discharge the water pressures are positive over the entire face of the dam. However, for a river flow of 1,200,000 cu ft per sec, it was found that a considerable part of the face is subjected to negative water pressures.

Using the pressures on the surface of the dam as obtained from these model tests and the severe assumptions of uplift ordinarily used for design purposes, it was found that the resultant of forces fell outside of the middle third of the base, which would produce tension at the heel. Field measurements of uplift pressures actually existing on the dam were made.[3] Using these data in place of the usual uplift assumption, it was determined that the structure as a whole was stable. The dam was poured in 5-ft lifts and 40-ft sections. Large stones

[3] "Uplift Measurements at Holtwood Dam," by Paul E. Gisiger, *Civil Engineering*, July, 1938, p. 447

were used as keys between the pours. In the event of extreme high flow, which would result in negative pressures of considerable magnitude acting on the top of the dam, weakness in any of the sections at this top-pour joint might cause the loss of some of the top slabs. Such a remote coincident would cause little damage, and the damage would be relatively easy to repair.

Comparison of Pier-Nose Design

Five different designs of noses were suggested for the Safe Harbor spillway piers, and tests were conducted to determine the most efficient design. The piers had already been designed before the tests were made with the emergency gate slot placed 4 ft 3 in. downstream from the nose of the dam. Since the piers were 8 ft 6 in. wide, a semicircular pier nose would be flush with the dam nose. For the first series of tests on pier noses, the emergency gate slot was fixed a model distance of $1\frac{9}{32}$ in., or a prototype distance of 4 ft 3 in., from the nose of the dam, and the pier-nose designs, which were longer, were permitted to extend beyond the dam nose. From this series of tests it was found that the semicircular pier nose was the most efficient.

Since the results of these tests proved contrary to the usual conception of streamlining, it was decided to make another series of tests with the pier nose flush with the dam nose in each case. To reduce the labor required to make the changes from one nose to the other, it was decided to eliminate both the gate and the emergency gate slots from the piers. The pier noses tested are shown in Fig. 13. As in Fig. 5, the coefficient of discharge was determined by

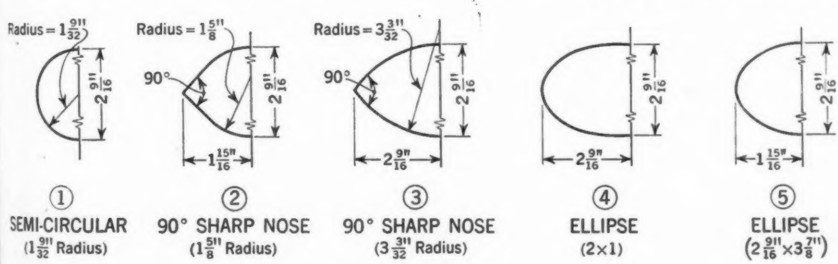


Fig. 13.—Pier Noses Tested

Eq. 1, L equals 2.392 ft; H equals one fortieth of the prototype head; and the approach area, in square feet, equals 2.819 (H + 0.838). The spillway section was the final design of Safe Harbor Dam—that is, a nose ellipse of 20 ft by 10 ft; a face curve expressed by $x^2 = 65\,y$; and a bucket radius of 42 ft. The results of this second series of tests are shown in Fig. 14. It will be noted that for this series of tests the semicircular design was the least efficient of the five designs tested. In the case of Design 2, Fig. 13, the pier nose was 2.3% less efficient at designed head when it extended $\frac{21}{32}$ in. beyond the dam nose than when it was flush with the dam nose. Design 4, Fig. 13, was 2.6% less

efficient when it extended $1\frac{9}{32}$ in. beyond the dam nose than when it was flush with the dam nose.

Since it was undesirable to move the gate slots farther downstream, and since there was ample discharge capacity provided without taking advantage of the 2.4% increase that would have been obtained if Design 3 had been used, it was decided to adopt the semicircular pier nose for Safe Harbor.

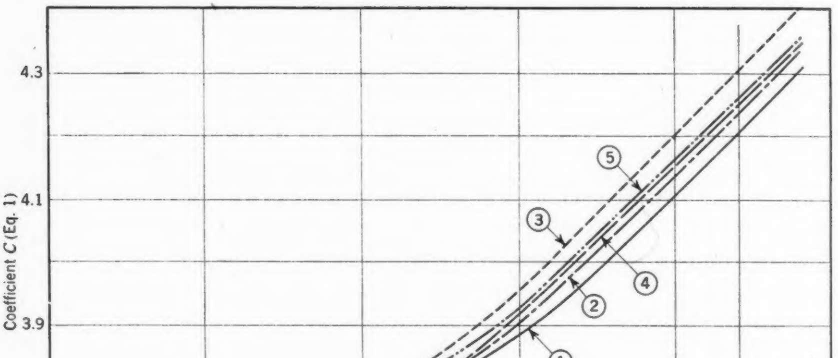


FIG. 14.—COMPARISON OF EFFICIENCY FOR VARIOUS PIER NOSE DESIGNS

The effect, on the various designs of pier noses, of flow approaching the piers from an angle was not investigated. It seems probable that the more streamlined pier noses would lose their advantage over the semicircular pier nose at relatively small angles of flow from the normal.

EROSION TESTS

When the erosion tests were made for the Safe Harbor development, there was no previous experience on a dam comparable to Safe Harbor that could be used as a starting point for this investigation. The normal head acting on the crest of Safe Harbor Dam is 32 ft, and the total drop from normal pond level to the bottom of the bucket is only 62 ft. With only one floodgate discharging, there is very little depth of water below the dam to act as a cushion pool.

Forty-four different designs of apron were tested, including stepped aprons, dentated sills, and aprons of various lengths and degrees of angular upturn.

Early in the program of tests it was found that the floodgate piers had to be extended at their full width to the downstream end of the apron to prevent erosion immediately below the structure where jets from adjacent gates met and seemed to be deflected downward with tremendous energy.

Of all of the types of aprons tested, only three gave adequate protection. One consisted of a 20-ft apron with a 42-ft radius starting from the bottom of the bucket, with a tangent extension of 12° from the horizontal to a step on the downstream end, which was 1 ft 8 in. high and 2 ft 6 in. wide. Erosion for this design started at a point 16 ft from the end of the apron and had a maximum depth of 32 ft at a distance of 128 ft. This design of apron was considered impractical since the Susquehanna River is full of debris at times of high flow, and it was felt that the step at the end of the apron would be damaged easily and would require frequent repairs. A 32-ft apron with a 42-ft radius starting from the bottom of the bucket, with a tangent extension at 16° from the horizontal and without a step, gave protection almost identical with the 20-ft apron with a step. Further tests were made with shorter aprons and steeper angular upturns. It was found that an 18-ft apron, with a 30-ft radius starting from the bottom of the bucket, with a tangent extension of 26° from the horizontal, resulted in erosion starting 20 ft from the end of the apron with a maximum depth of 47 ft at a distance of 140 ft from the end of the apron (see Fig. 15). This apron was adopted for Safe Harbor.

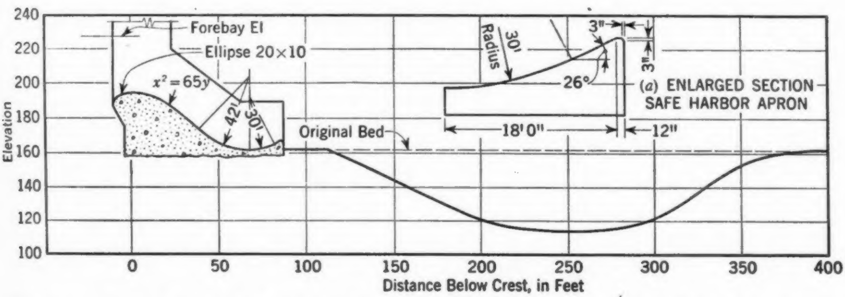


FIG. 15.—ULTIMATE EROSION FOR ANY COMBINATION OF GATES DISCHARGING AT SAFE HARBOR DAM

The jet of water leaving the 32-ft apron was smooth on the surface until it hit the standing wave that was formed downstream. The jet of water leaving the 18-ft apron was very turbulent, indicating that the lower parts of the jet were abruptly deflected upward into the upper part of the jet.

Since the starting point of erosion was substantially the same distance downstream from the end of the apron for both the 32-ft apron and the 18-ft apron, it is evident that the angle with which the jet leaves both aprons must be approximately the same. Undoubtedly this same angle could be obtained with a variety of combinations of length and upturn. However, tests on aprons even slightly shorter than 18 ft, with steeper angles of upturn, resulted in an unstable jet.

After the experience obtained by the erosion tests on the Safe Harbor model, it was a simple matter to arrive at an appropriate design of apron for the

Holtwood Dam. The erosion conditions at Holtwood are far less severe than at Safe Harbor, since there are no gates on this dam and, as the head increases on the dam, the tailwater level builds up, forming a cushion pool that absorbs most of the energy. The proposed design of apron for Holtwood that was developed from these tests is shown in Fig. 16. It will be noted that this apron

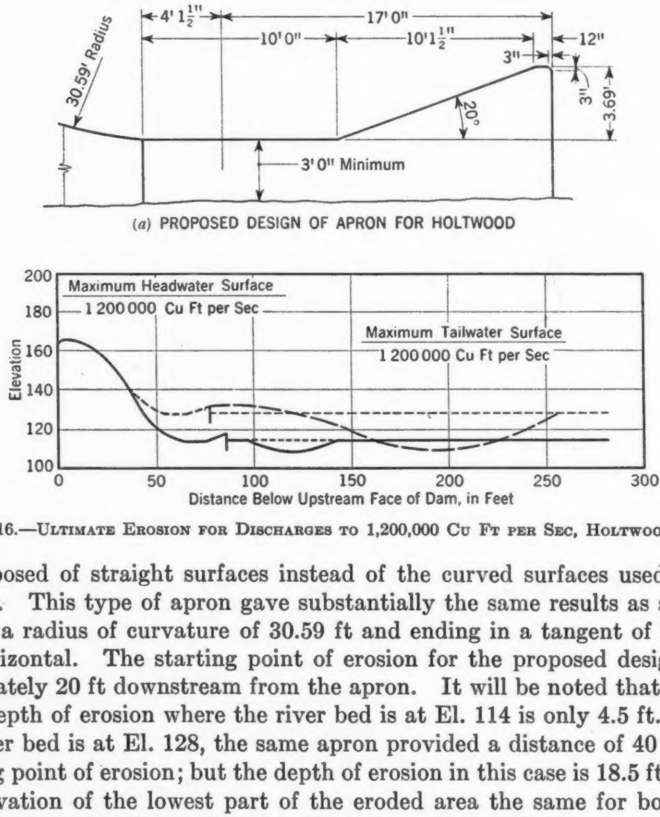


FIG. 16.—ULTIMATE EROSION FOR DISCHARGES TO 1,200,000 CU FT PER SEC, HOLTWOOD DAM

is composed of straight surfaces instead of the curved surfaces used at Safe Harbor. This type of apron gave substantially the same results as an apron having a radius of curvature of 30.59 ft and ending in a tangent of 20° from the horizontal. The starting point of erosion for the proposed design is approximately 20 ft downstream from the apron. It will be noted that the ultimate depth of erosion where the river bed is at El. 114 is only 4.5 ft. Where the river bed is at El. 128, the same apron provided a distance of 40 ft to the starting point of erosion; but the depth of erosion in this case is 18.5 ft, making the elevation of the lowest part of the eroded area the same for both cases. Since the water surface below the dam is at substantially the same elevation across the entire river, the depth of the cushion pool for ultimate erosion in both cases is the same, as might be expected. It is interesting to note that, for the tests with the river bottom at El. 128, some of the gravel was carried upstream and deposited on the original bed.

DEFLECTION-WALL TESTS

The Holtwood development was built at the upstream end of an island. This island with one immediately downstream divides the river into two channels for a distance of one mile. There is considerable drop in water-surface levels in this distance. The channel on the east side of these islands is now used as the power plant tailrace, and the main river channel is on the

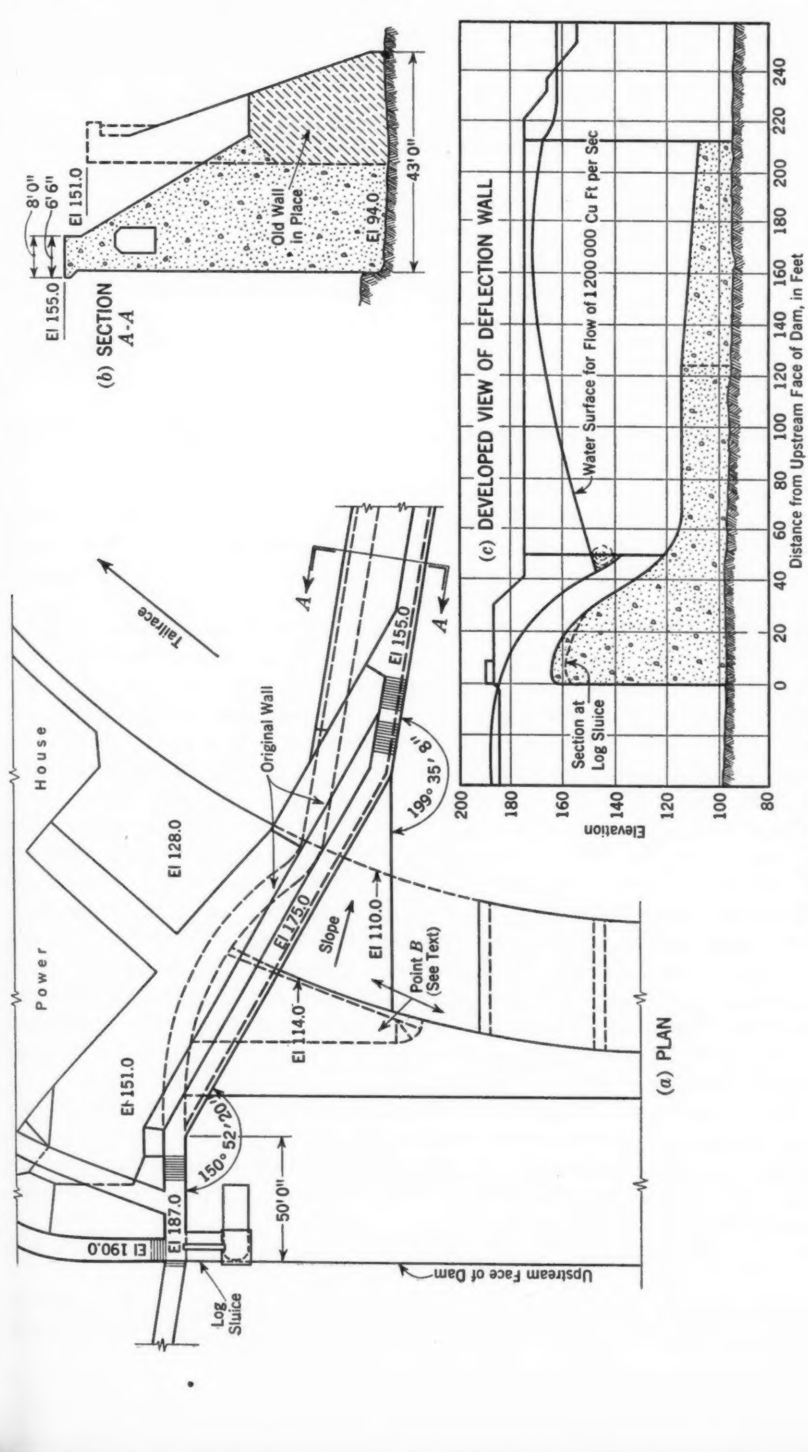


FIG. 17.—TAILRACE DEFLECTION WALL, HOLTWOOD DAM

west side of the islands. A deflection wall was built from the east abutment of the dam to the head of these islands so that, in times of high flow, water going over the dam would be diverted to the west channel, thereby preventing a large increase in tailrace elevation. The shape of the original deflection wall is shown in Fig. 17. The curved part of this wall, which protected the end of the power house, was a rather massive structure. The straight extension between the tailrace and the main river channel was a relatively light section. It will be noted that the curved section of the wall cuts across in front of the spillway. At the peak of the 1936 flood, the water piled up behind this wall to such an extent that the straight section was overtopped and failed.

The purpose of the deflection-wall tests was to arrive at a layout for the part of this wall, which cuts across in front of the spillway, that would deflect the flow with a minimum height of wave along the wall. It was felt that such a layout would be the most economical as well as the most effective.

Since it was undesirable to reduce the width of the tailrace, the straight section of the new wall, Fig. 17, was fixed in the location shown. It was also necessary to maintain a certain minimum cross section of the transition wall that connects the east dam abutment with the straight section of wall. Numerous layouts of this transition wall were tested, and the one shown in Fig. 17 was finally adopted. The height of wave along this wall for a discharge of 1,200,000 cu ft per sec is also shown.

A construction bridge with top elevation of 128 was in place as shown in Fig. 17(*a*). With the old layout a block of concrete had been poured as shown, sloping downward from this construction bridge to El. 114, and the space between this and the toe of the dam was paved (see point *B*, Fig. 17(*a*)). It was found necessary to remove this sloping section and two arches of the construction bridge in order to hold the wave to a reasonable height along the wall.

There was some concern regarding the force of impact acting on the transition section of the wall. Consequently, sixteen piezometers were placed to measure these forces. The first set of piezometers was placed 10 ft (prototype) downstream from the starting point of the transition section, and the others were spaced uniformly along its full length. In no case was the pressure obtained by a piezometer greater than the pressure computed from the height of the water surface at that point, and in most cases it was appreciably less. From this it may be concluded that the zone of impact is rather limited and the wall outside of this zone is protected from impact by the water that has already been deflected.

From these tests it was evident that a straight transition section was superior to any curved section that would attempt to eliminate or at least reduce the first shock. The problem of deflecting a part of a stream over into the main body is evidently quite different from the problem of deflecting the entire jet as is done by the bucket of a dam.

Conclusions

It is felt that, of the four spillway designs tested, Design 2 is to be preferred. Although the section with the nose designed according to Mr. Winter's test

data is about 1% more efficient at designed head, it is probable that negative pressures exist somewhere on the spillway face that might cause damage to the concrete surface.

It was a bit disconcerting to have such a marked difference in the characteristics of the discharge coefficient curves with the velocity of approach correction for the two series of tests. The projecting nose of the Safe Harbor Dam seems to offer the only logical explanation.

The results of the tests with the upstream cofferdam cribs in place are valuable in that there are, no doubt, many instances where there would be a considerable saving of expense by leaving these cribs in place.

The measurements of pressure on the spillway face agreed with what had been anticipated. The check that was obtained on the Safe Harbor spillway with pressures observed on the model agreed within reasonable limits.

The results of the comparative tests on pier noses were most enlightening. It is evident that it is very important from an efficiency point of view to have the pier noses and the dam nose in the same plane.

It is felt that the apron designs that were developed are the most economical possible under the existing conditions in each case. It seems logical that the energy of the water should be utilized to make the starting point of erosion downstream as far as possible from the end of the apron, rather than to attempt to dissipate the energy on the apron.

The tests made to find a suitable design and location for the Holtwood deflection wall were not unusual in any way. Many such problems have been solved in a similar manner and, until the knowledge of hydraulics increases, future problems of a like nature may be solved satisfactorily by means of model tests. It was evident from these tests that a straight transition section was preferable to any other design and that the angle across the flow should be kept as small as possible. The tests showed that the zone of impact was confined to a distance of less than 10 ft from the starting point of the transition wall.

Acknowledgments

Both series of tests reported in this paper were made at the Alden Hydraulic Laboratory of the Worcester Polytechnic Institute, at Worcester, Mass. The testing was under the supervision of C. M. Allen, M. Am. Soc. C. E. The writer wishes to express his appreciation to all members of the laboratory staff and in particular to Professors Allen, L. J. Hooper, and C. W. Hubbard. Credit is also due J. Marcus Mousson, 2d, M. Am. Soc. C. E., of the Safe Harbor Water Power Corporation for his assistance in this work.

A M

initi

of r
fabr
of
stru
pro

mit
pro
for
pu

to

str
in

C
re
an
re

r
p

y
u

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

REPORTS

DESIGN OF STRUCTURAL MEMBERS

FIRST PROGRESS REPORT OF THE COMMITTEE OF THE STRUCTURAL DIVISION ON DESIGN OF STRUCTURAL MEMBERS

Introduction

In this Report the objectives of the committee will be outlined and the initial progress reviewed.

The increasing use of a wider variety of structural alloys, the introduction of new types of structural design, and the development of new methods of fabrication—these are principal factors that require periodic reconsideration of structural design methods along the following lines: Consideration of structurally significant properties of new materials; and correlation of these properties with structural design procedures.

The Executive Committee of the Structural Division organized this committee in January, 1940, with the authorized purpose "To consider the physical properties of metals and their influence on the design of structural members for resistance to static, dynamic, and pulsating loads." In line with this purpose the committee has undertaken two main projects:

(1) A study of the physical properties of four structural alloys, in relation to their behavior under varying conditions of stress and shape; and

(2) A study of the design of compression members, including the design of structural alloy columns and the design of plate elements stressed so as to involve questions of structural stability.

The present status of these two projects will be reviewed in this Report. Complete bibliographies relating to these projects will not be attempted but references will be made to some principal sources of information. References are not necessarily original sources but have been selected on the basis of readability and in some cases for their extensive bibliographies.

Project 1. The Structural Properties of Materials

This project is concerned with the physical properties of materials and the relation of these properties to the behavior of structural members. The physical properties of a material, as usually evaluated by a tensile test, are the

Note.—This Report was presented in abstract form at the Annual Meeting of the Society, New York, N. Y., on January 22, 1942. Written comments are invited for immediate publication; to insure publication the last discussion should be submitted by August, 1942.

yield strength, ultimate strength, percentage elongation at fracture, and percentage reduction in area. One of the motivating factors in originating this project was the limitation in certain specifications that the yield point of alloy steels be set at not more than 70% of the ultimate strength (1).[1] Such a specification indicates the desirability of knowing more about the structural significance of the shape of the stress-strain diagram as determined in the usual tension test (2)(3)(4).

In an actual structure the principal stresses no longer act in only one direction, as in a tension test, but may exist in any direction, in any combination, and may vary more or less rapidly in direction and magnitude from point to point, depending on the shape and load characteristics. Properties, such as ductility and brittleness, as evidenced in a standard tension test, will not necessarily be the same in a state of combined stress. Ductile yielding in a tension test results from the multiple slipping of adjacent crystal planes and is caused by the shear stresses which are maximum on planes 45° from the plane of maximum tension. Brittle fracture, on the other hand, is associated with a direct tearing apart of the grain structure, and will occur when (and if) the tensile stresses exceed the cohesive strength of the material before the maximum shear reaches the critical flow stress required to produce ductile yielding. The ratio of maximum shear stress divided by maximum tension stress will give an indication of the tendency toward ductile or brittle behavior in any combined state of stress. In the simple tension test this ratio is $\frac{1}{2}$. If the ratio is more than $\frac{1}{2}$ in a state of combined stress the material may be expected to exhibit more ductility than it does in an ordinary tension test. On the other hand, if the combined stress state is predominantly tension in more than two directions, the $\frac{\text{shear}}{\text{tension}}$ stress ratio may be greatly reduced.

In such states of multiaxial tension stress, two samples of metal which are identical in ductile properties in the usual tension test may exhibit opposite types of behavior, one ductile and the other brittle, depending on their relative cohesive strengths. In a state of equal tension in every direction no shearing stress would exist, and ductile yielding would be impossible. Although it is impracticable to make tests in a complete state of equal triaxial tension, borderline states of stress, for example, in deep notches (5) may result in a marked increase in the tendency toward brittle behavior. S. L. Hoyt (6) states that notch bar impact tests show principally the "notch sensitivity" of a material. "Notch sensitivity" may be defined as susceptibility to brittle failure caused by localized regions of multiaxial tensile stresses. Low temperatures increase the "notch sensitivity" of some steels (7).

Most engineers are familiar with cases in which structural parts have failed in a brittle manner under increasing static load, yet the material may show good ductility in a simple tension test. The matter is one of particular concern in the welding industry where brittle fractures in heat affected zones, in some cases, have been thought to be caused by multiaxial stresses induced by shrinkage, external load, or both (8).

[1] **Numerals in parentheses, thus: (1), refer to the corresponding references in the Bibliography (see Appendix).**

The behavior of a metal under combined stress is only one of many factors that may act to determine the general behavior and structural usefulness of a structural member under load (9). In the actual structure, the stresses not only are combined, but the magnitude and orientation of the combined stresses may vary more or less rapidly from point to point. Additional factors involving shape of cross section, distribution and intensity of stress concentrations and load, change in structural behavior under load, rate of application of load, temperature, corrosion, fatigue, questions of stability—any one or all of these—may affect the structural usefulness of a member during its life history. The entire problem is one in which general conclusions are usually impossible and any particular type of structural member must be considered in connection with the material from which it is made and the use for which it is intended.

The study of some of the foregoing problems had been undertaken by the former Sub-Committee No. 1 (Design Stresses in Relation to Ultimate Strength, Yield-Point Strength, and Related Physical Properties of Metals) of the Structural Division Committee on Fundamentals Controlling Structural Design. In continuing this work the following steps were proposed:

(1) Study the literature and various theories relating to the strength and physical properties of metals under combined stress;

(2) Make a survey of those metals in current structural use, and select typical materials for further test; and

(3) Conduct a series of tests on each metal under a multiplicity of different stress and shape conditions and compare the behavior in these tests with tension test characteristics.

The facilities of the Fritz Engineering Laboratory at Lehigh University, Bethlehem, Pa., were offered to the committee, including the services of a half-time research fellow for a period of at least two years. Donations of a wide variety of alloy steel samples were received from the U. S. Steel Corporation and the Bethlehem Steel Company. Preliminary tensile tests were made on these steels and a tentative program of tests prepared. After study the program was revised and adopted by the committee in January, 1940. Four materials were selected for test: Carbon structural steel, silicon structural steel, a typical low-alloy structural steel, and structural aluminum alloy 27ST.

In January, 1940, the former sub-committee conducting this program ceased to exist, and the work was taken over as Project 1 of the present committee. Plate material for the tests was obtained, the structural aluminum alloy being donated by the Aluminum Company of America. The Chicago Bridge and Iron Company donated $450 as a fund for the purchase of materials and equipment.

The program originally planned is now 50% complete, but no research fellow is available for the continuation of the work at the present time. A part of the program includes pinned and riveted connection tests similar to tests reported by Leon Moisseiff, M. Am. Soc. C. E. (10), but these differ sufficiently in detail so as to supplement rather than duplicate the tests made at the Aluminum Research Laboratories. These tests have now been com-

pleted by Lloyd Green, Jun. Am. Soc. C. E., former Research Fellow at the Fritz Laboratory.

At a meeting of the American Society for Testing Materials (A. S. T. M.) on June 26, 1940, a "Symposium on the Significance of the Tension Test of Metals in Relation to Design" was held (4). This committee was invited to attend and take part in the symposium. The results of this A. S. T. M. symposium were of direct use to the work of this committee, reference being made in particular to the paper by C. W. MacGregor (4) in which improved methods of evaluating the ductility aspects of the standard tension test are presented.

Early in 1940 Project 1 was expanded to include a survey of the compressive stress-strain characteristics of a large number of samplings of structural alloys. This was to be correlated with work the committee had undertaken on the design of compression members. Accurate knowledge of compressive stress-strain relations is a fundamental factor in the design of structural elements loaded primarily in compression. Donations of steel from different heats and different rollings were received from the Bethlehem Steel Company, Otis Steel Company, U. S. Steel Company, and the Youngstown Sheet and Tube Company.

These samples included eleven different heats of carbon structural steel, seven of silicon structural steel, fifteen of various low-alloy high-strength steels, and two samples of structural aluminum alloy 27ST. At least four different tests were made from each sample, these being tension and compression tests both with and across the direction of rolling. As much information as practicable was obtained from each test, including (in the case of steel) both upper and lower yield points, Young's modulus, Poisson's ratio, and both "uniform" and "necking" reduction in area, as suggested by Mr. MacGregor (4). Uniform reduction in area occurs along a member of uniform cross section and uniform tensile stress up to the maximum load. The load falls off during necking or localized reduction. Two materials of identical composition and identical percentage elongation in a short gage length may have entirely different ratios of uniform-to-necking reduction in area, depending on the process of manufacture. The possible importance of such differences may be illustrated by comparing the over-all elongation of two long wires, stressed in tension, one of which has mostly uniform ductility and the other primarily necking ductility. Short samples of each wire might conceivably show the same percentage elongation in 2 in. in a laboratory tension test. In a long length, the wire with primarily uniform ductility would stretch considerably before final fracture, but the wire with primarily necking ductility would fracture with an insignificant amount of total elongation.

The following conclusions from the report on this work relate in particular to the design of compression members:

(*a*) The average values of the upper and lower yield points of each type of steel tested were practically the same in compression as in tension; and

(*b*) The general shape of the individual stress-strain curves up to the upper yield point and into the lower yield point range were very nearly alike in tension and compression.

The report also gives typical stress-strain curves for each material showing both the most linear and least linear stress-strain diagrams up to the yield point for each type of steel. The shape of the stress-strain curves is of basic importance in compression member design.

The status of Project 1 may be summarized as follows: The original program of comparative tests of structural alloys is about 50% complete; and an additional program of compression and tension tests of thirty-five samples of structural alloy has been completed and reported in detail in the 1941 *Proceedings* of the A. S. T. M. (11).

Project 2. Design of Structural Alloy Columns

In January, 1940, the committee was asked to study a memorandum on the subject of "Unit Stresses in Columns of High Yield-Point Steels" which had been prepared by Jonathan Jones, M. Am. Soc. C. E., in October, 1939. As a result of this study the committee reported that:

(*a*) More knowledge was needed on crookedness of columns as actually built;

(*b*) Formulas for eccentrically loaded columns required further study;

(*c*) The importance of the compressive stress-strain characteristics of the material was emphasized, and the need for further work in this field was outlined;

(*d*) The selection of reduced l/r for columns acting as a part of frames was a subject requiring further study; and

(*e*) The most important and most difficult question to decide upon is the "factor of safety" to be used in design formulas.

The suggestion was made by Mr. Jones that the committee broaden the scope of its column studies along these lines. This suggestion was approved by the Executive Committee, and this became Project 2 for the committee.

The suggestion listed as Item (*c*) resulted in the program of compression tests of structural alloys which has been discussed as a part of Project 1.

Under Item (*a*)—a study of the crookedness of columns as actually built—it was found to be impracticable to survey columns in fabrication shops; nor would such information provide the desired answer as to the columns as actually constructed, since columns may be bent between shop and site of erection. Various inspection agencies were contacted and asked the question: "Could you state that the crookedness of steel columns as erected is always less than the rolling mill tolerances (for straightness) specified by the American Institute of Steel Construction (A. I. S. C.)?" The replies appear to be based primarily on opinion, but the general feeling among these agencies is that when buildings are erected and riveted, the columns are straight to a greater degree than is required by the A. I. S. C. tolerances. The surveys made by certain of the agencies among their branches in all parts of the United States were quite complete and the inquiry at least has served the purpose of focusing some attention on the importance of column straightness.

Additional information on the subject of column straightness is being furnished to this committee by Prof. L. T. Wyly, M. Am. Soc. C. E., of North-

western University, Evanston, Ill., who will report at a later date on a number of field measurements on columns in structures.

Several research projects sponsored at Lehigh University by A. I. S. C., are also correlated with the work of this committee. These include tests of both axially and eccentrically loaded I-section columns, tests of columns built into a frame, and buckling tests of plates in compression stiffened by horizontal and vertical stiffeners.

A preliminary report covering the tests of eccentrically loaded columns and including a general study of steel column formulas was circulated to the committee in January, 1941.

This committee has not lost sight of the very extensive work done by two former Society committees. The first of these was the Special Committee of the Board of Direction on Steel Columns and Struts, organized in January, 1909, and whose work was terminated in 1918. Special reference is made to the "General Programme for Column Tests" in a closing discussion of this committee's final report (12).

This program, with minor modification, might well serve today as a guide to this committee on the question of further column research. As a matter of fact, the work now in progress or recently completed fits into the program very well.

In 1923 another Society committee, the "Special Committee of the Board of Direction on Steel Column Research" was formed. It submitted three reports (13)(14)(15) and finished its work in 1933. These reports cover a very complete and detailed review of previous column tests together with results of many new tests made for the committee.

Other important sources of information have been studied (16)(17)(18). In addition to providing sources of information, these include references to most of the work done on columns during the past two hundred years.

A summary of the factors that affect the strength of a column will provide the basis for understanding the problem undertaken by the committee. These factors may be defined by the way they affect the strength of an "idealized" column. An "idealized" column will be defined as one that is made of a perfectly elastic material, is loaded axially through frictionless pins at each end, is perfectly straight, and does not fail locally. This idealized column will buckle elastically at the "Euler" critical load

$$P_{cr} = \frac{\pi^2 E I}{l^2} \qquad (1)$$

in which: E = Young's modulus, I = moment of inertia, and l = length of column.

The idealized column is usually quite different from the actual column as constructed and used in a structure. In the actual column the strength of the column is different from that given by the Euler formula because of: (1) The non-linear shape of the stress-strain relation, (2) accidental imperfections, (3) the known end eccentricity, (4) the shape of the cross section, (5) the torsional behavior, (6) shearing deformation, (7) local buckling or

crippling of a part of the column, (8) method of fabrication, and (9) continuity of action in a frame.

(*1*) *Non-Linear Shape of Stress-Strain Relation.*—Above the proportional limit Young's elastic modulus is no longer a constant. The strength of the column is reduced, and may be determined approximately by using a "reduced modulus," E_R, in the "Euler" formula, Eq. 1 (19). In the case of structural steels the yield point represents the practical upper limit of column strength for short columns which do not buckle elastically.

(*2*).—Accidental imperfections, such as curvature, end eccentricity, non-homogeneity, etc., act to reduce the strength.

(*3*) *Known End Eccentricity.*—When the material has an elastic stress-strain relationship the maximum stress may be calculated by the "secant" or "eccentricity" formula. In the case of materials with a well-defined yield point, such as structural steel, the load at which maximum stress reaches the yield point may be divided by an arbitrary factor of safety to indicate a safe design load. When the eccentricity is in the strong plane the possibility of lateral buckling should be investigated (20).

(*4*) *Shape of Cross Section.*—The shape of the column cross section affects the strength when considered in conjunction with a material having a non-linear stress-strain relation (19).

(*5*) *Torsional Behavior.*—Certain shapes of thin material may buckle by twisting, under either axial or eccentric load (21).

(*6*) *Shearing Deformation.*—The theoretical strength of a column is reduced, especially in the case of the built-up column, when shearing deformation is considered.

(*7*) *Local Buckling or Crippling of a Part of the Column.*—Many different cases are reviewed by S. Timoshenko (17).

(*8*) *Method of Fabrication.*—A method of fabrication which introduces initial stresses or causes warping of the component parts may reduce the strength of the column.

(*9*) *Continuity of Action in a Frame.*—Compression in a strut reduces its bending stiffness, whereas tension increases the bending stiffness. Buckling of a member in a frame ensues when the summation of bending stiffness becomes equal to zero at any joint of a frame (22).

It is also important to emphasize that much of the knowledge of column behavior has been based on laboratory experiments in which the ends of the column are either milled flat, or simulate a pin end by use of a knife edge or a roller nest. Actual columns usually have framed end connections which are not equivalent to the end conditions in the usual laboratory test. In a laboratory test of an eccentrically loaded column the eccentricity is usually maintained at a constant value up to failure, but the equivalent eccentricity of load in a framed column varies as the load varies. For these and other reasons the differences between laboratory tests and actual column behavior should always be kept in mind.

The multiplicity of factors affecting column strength has led to some confusion of thought in dealing with the problem. It is obviously impracticable to consider all factors at once in a design formula. Investigators frequently

have considered only one or two of the factors and have then magnified the factors considered to include arbitrarily all of the others. For example, in the case of non-ferrous alloys and some of the high-strength steels, the non-linear stress-strain relationship (19) may well be the most important factor affecting the strength of an axially loaded column. Imperfections as to shape, curvature, and accidental eccentricity may be covered approximately by modifying the assumed stress-strain relationship. In the case of structural steel, accidental eccentricities and curvature may be the more important factors, and the relatively small variation from a linear stress-strain relation up to the yield point may be taken care of by modification of the eccentricity or secant formula. Another investigator (23) has proposed to take account of all factors by assumed initial curvature of the column, which results in formulas for maximum stress similar to the secant formula.

Mention should also be made of the paper by Leon S. Moisseiff and Frederick Lienhard, Members, Am. Soc. C. E. (24), which proposes rules of design for the plate elements in compression members.

Aside from this preliminary review of literature and statement of problems and objectives, the committee is not as yet prepared to present recommendations or report on progress until further study and research now under way are completed. The committee proposes to sponsor papers on topics related to its work and will attempt to bring together conflicting ideas about column design formulas and propose new research projects to organizations, such as the A. I. S. C., interested in sponsoring such work.

Respectfully submitted,

Hardy Cross
F. H. Frankland
H. J. Gilkey
E. C. Hartmann
H. D. Hussey
W. H. Jameson
C. T. G. Looney
Milton Male[2]
Joseph Marin
L. S. Moisseiff
N. M. Newmark
William R. Osgood[2]
J. W. Shields
F. P. Thomas[2]
R. L. Templin
H. E. Wessman
G. B. Woodruff
L. T. Wyly[2]

Bruce Johnston, *Chairman*

April 15, 1942

[2] Appointed a member of the committee on January 29, 1942.

APPENDIX

Bibliography

Project 1.—

(1) **A. R. E. A. Specifications for Steel Railway Bridges,** 1938, Pt. 3, Article 9.

(2) **"Theories of Strength,"** by A. Nádai, *Journal of Applied Mechanics,* A. S. M. E., Vol. 1, No. 3, July, 1933, pp. 111–129. Nádai reviews the development of the most widely accepted theory for the yielding of metals.

(3) **"Strength of Metals Under Combined Stress,"** by M. Gensamer, Am. Soc. for Metals, 1941. Includes prevailing notions regarding cohesive failure and is easy to read.

(4) **"Symposium on Significance of the Tension Test of Metals in Relation to Design,"** by J. M. Lessels, C. W. MacGregor, Fred B. Seely, H. W. Gillett, and discussers, *Proceedings,* A. S. T. M., 1940, pp. 501 to 609. These papers provide fairly complete Bibliography as well as valuable information.

(5) **"Kerbspannungslehre,"** by H. Neuber, Julius Springer, 1937. Stress distribution in the vicinity of deep notches.

(6) **"Notched-Bar Testing and Impact Testing,"** by S. L. Hoyt, *Proceedings,* A. S. T. M., Vol. 38, 1938, Pt. II, pp. 141–156.

(7) **"Impact Resistance and Tensile Properties of Metals at Subatmospheric Temperatures,"** by H. W. Gillett, Project No. 13 of the Joint A. S. M. E.–A. S. T. M. research committee on effect of temperature on the properties of metals. Excellent Bibliography and review of impact tests at low temperatures.

(8) **"Weldability—Cracks and Brittleness Under External Load,"** by W. Spraragen and G. E. Claussen, *Journal,* Am. Welding Soc., *Research Supplement,* August, 1941, p. 340-S. A review of the literature on this subject.

(9) **"The Structural Significance of Stress,"** by Bruce Johnston, *Civil Engineering,* May, 1939, pp. 291–294. General article on factors affecting the structural usefulness of a structural member.

(10) **"Comparative Tests of Riveted and Pin-Connected Joints of Steel and Aluminum Alloys,"** by L. S. Moisseiff, E. C. Hartmann, and R. L. Moore, presented at the Annual Meeting of the American Society of Civil Engineers in January, 1942.

(11) **"Compression and Tension Tests of Structural Alloys,"** by Bruce Johnston and Francis Opila, *Proceedings,* A. S. T. M., Vol. 41, 1941, pp. 552–578.

Project 2.—

(12) **Final Report of the Special Committee on Steel Columns and Struts,** *Transactions,* Am. Soc. C. E., Vol. LXXXIII (1919–1920), pp. 1584–1688.

(13) **Progress Report of the Special Committee on Steel Column Research,** *Transactions*, Am. Soc. C. E., Vol. 89 (1926), p. 1485.

(14) **Second Progress Report of the Special Committee on Steel Column Research,** *ibid.*, Vol. 95 (1931), p. 1152.

(15) **Final Report of the Special Committee on Steel Column Research,** *ibid.*, Vol. 98 (1933), p. 1376.

(16) **"Columns,"** by E. H. Salmon, Oxford Technical Publications, 1921.

(17) **"Theory of Elastic Stability,"** by S. Timoshenko, McGraw-Hill Book Co., Inc., New York and London, 1936.

(18) **First, Second, and Final Reports of the Steel Structures Research Committee,** Dept. of Scientific and Industrial Research, Great Britain, 1931, 1934, 1936.

(19) **"Column Curves and Stress-Strain Diagrams,"** by William R. Osgood, *Research Paper No. 492*, Bureau of Standards, October, 1932.

(20) **"The Lateral Buckling of I-Section Column with Eccentric End Loads in Plane of the Web,"** by Bruce Johnston, *Journal of Applied Mechanics*, A. S. M. E., December, 1941, Vol. 8, No. 4, p. 176.

(21) **"Torsional and Flexural Buckling of Bars of Thin-Walled Open Sections Under Compressive and Bending Loads,"** by J. N. Goodier, presented at Annual Meeting of the American Society of Mechanical Engineers, December 2, 1941.

(22) **"Principles of Moment Distribution Applied to the Stability of Structural Members,"** by Eugene E. Lundquist, *Proceedings*, Fifth International Cong. for Applied Mechanics, 1938, pp. 145–149.

(23) **"Rational Design of Steel Columns,"** by D. H. Young, *Transactions*, Am. Soc. C. E., Vol. 101 (1936), pp. 422–500.

(24) **"Theory of Elastic Stability Applied to Structural Design,"** by Leon S. Moisseiff and Frederick Lienhard, *ibid.*, Vol. 106 (1941), pp. 1052–1112 (contains an extensive Bibliography).

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

MODEL TESTS ON STRUCTURES FOR HYDROELECTRIC DEVELOPMENTS

Discussion

By Messrs. Emil P. Schuleen, and C. M. Allen

Emil P. Schuleen,[4] Assoc. M. Am. Soc. C. E.[4a]—Holtwood Dam (as described by Mr. Davis) was originally designed for a maximum flood of 725,000 cu ft per sec. During the flood of 1936 this flow was exceeded by about 20% when a discharge of some 860,000 cu ft per sec occurred. It is interesting to note that the subsequent model tests showed the design of the spillway to be safe, with only slight negative pressures on the spillway face, even for the new design requirements for a flood of 1,200,000 cu ft per sec, an increase of about 66% over the original assumptions for maximum flow. It is fortunate that the original design was sufficiently liberal to provide this factor of safety.

The erosion tests conducted during the investigations deserve comment. The reproduction in a model of the resistance of the stream-bed material to scour cannot be made with absolute fidelity by any of the usual laboratory methods. Such tests give qualitative rather than quantitative results, indicating the points of maximum and minimum erosion, and to a limited extent the general location of the deposition of bed materials. The use of a movable bed in the model, even if it consists of gravel or stone sizes closely simulating the size of material found in the prototype, can be considered to indicate only the relative scouring potentialities for the various designs tested. Caution must be exercised, therefore, in the interpretation of the results of erosion tests, which should not be considered to indicate the exact amount of scour to be expected. In this connection, it is also advisable in most instances to obtain data of a negative nature, whereby the investigation would show the behavior of the structure on the assumption that the resistance of the bed material to erosion and movement is much greater than anticipated and that a negligible amount of scour would occur in the prototype. This can be accomplished by the use of a fixed bed in the model. Since the degree of

Note.—This paper by L. M. Davis, Assoc. M. Am. Soc. C. E., appears on pp. 543–564 of this issue of *Proceedings*.

[4] Senior Engr., U. S. Engr. Office, Pittsburgh, Pa.

[4a] Received by the Secretary March 2, 1942.

erosion is more or less unpredictable, no possibility should be overlooked in investigations of this type.

The tests to determine the crest profile for the Safe Harbor spillway emphasize the delicate relation between the shape of the spillway at the crest and its effect on discharge efficiency. In general, the coefficient of spillway discharge is dependent on the shape of the spillway crest and the head. If the spillway crest is designed properly, the pressure distribution in the vicinity of the crest will be zero only at the design head. Under this condition the crest profile would be that of the lower surface of the nappe of an equivalent free-flowing jet, and a coefficient of discharge of approximately 4.0 would result. For heads lower than the design head, positive pressures would be produced and the coefficient decreased. Conversely, the reduction in pressure on or in the region of the spillway crest, due to heads in excess of the design head, would result in an increase in the discharge coefficient. It is interesting to note that slight revisions in the shape of the crest profiles tested (some seemingly insignificant) produced appreciable changes in the coefficient of discharge.

The reason for the increase in the discharge coefficient for the Safe Harbor spillway, with decrease in the depth of approach due to silting, is not apparent. The opposite is true of the Holtwood spillway, which is in conformity with results obtained by others in model research and in analyses of actual structures. Inspection of both designs reveals that the Safe Harbor spillway differs from Holtwood only in two essential respects—the use of the overhanging lip upstream from the crest, and the presence of gate piers on the spillway section. This would suggest that the increase in the discharge coefficient with decrease in the approach depth may be due either to the effect of the pier noses and the gate recesses, or to a redistribution of flow at higher velocities at a point immediately upstream from the crest. The latter seems to be the more logical answer to the problem. The discharge coefficient, under normal conditions, is practically the same for a spillway crest with an overhanging lip as it is for a crest with a vertical upstream face. The effect of a reversal in flow lines produced by the overhanging lip apparently is compensated for by an increased curvature of the filaments adjacent to the spillway face. However, with increased velocity upstream due to a decrease in the approach depth, it is possible that this compensation does not occur, resulting in a different condition of flow than before and producing an alteration in the shape of the flow profile over the crest of the spillway. However, this is merely a conjectural solution to a problem which can be solved only by intensive research study.

The apron designs for both the Safe Harbor and Holtwood dams are unusual in that no direct dissipation of energy is intended on the apron proper. The purpose of each apron is to deflect the high velocity jet away from the structure and let it dissipate itself in impact as it strikes the tailwater and stream bed below the end of the apron. The model investigations were for the purpose of determining the relative scouring effects and of producing a design whereby the resulting scour would be limited to areas a safe distance from the toe of the dam.

The hydraulic performance of an apron is a function of the head, the velocity and thickness of the issuing jet, and the tailwater conditions available. Since these factors differ widely for each case under consideration, the design details must be such as to fit the physical and hydraulic characteristics of each particular case. The performance must be stable and of such character as to produce no serious damage to the structure or its foundation. These requirements apparently have been met in the case of the Safe Harbor and Holtwood spillways, since each has weathered floods of great magnitude without serious consequence. There is no question, therefore, with respect to the adequacy of the design in these two cases.

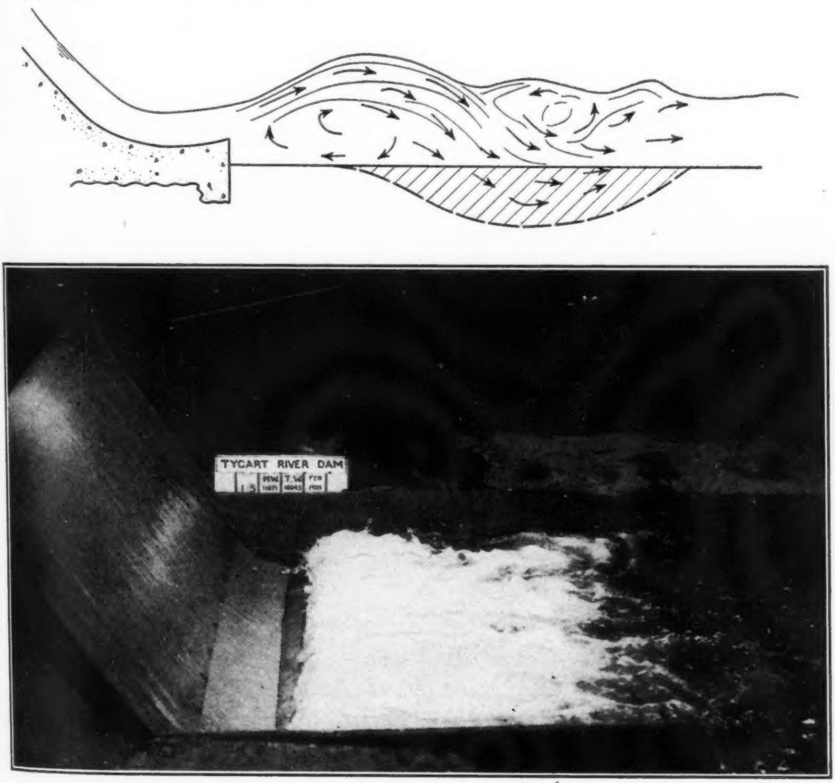


Fig. 18.—Flow with Ground Roller

There is a tendency among engineers to avoid the expense and time required for model tests by applying the results of tests of other structures to their problem. The practice of making use of all available information on the subject should not be discouraged, but the engineer should be cautioned to make certain that the test results definitely are applicable in his case. In this respect, the general type of apron used for the Safe Harbor and Holtwood dams has certain operation characteristics that would be difficult, if not

impossible, to predict except by hydraulic experiments on a model of the particular structure to which the design is to be applied. Studies made on aprons of this general type in the course of the model tests for the Tygart River Dam (in West Virginia) revealed that two distinctly different types of flow can exist, depending on the depth of tailwater available. When the tailwater is sufficiently low, the jet will be deflected upward by the end of the apron, and dissipation of energy will occur by impact of the jet with the tailwater and the stream bed, as indicated by Fig. 18. This type of action is

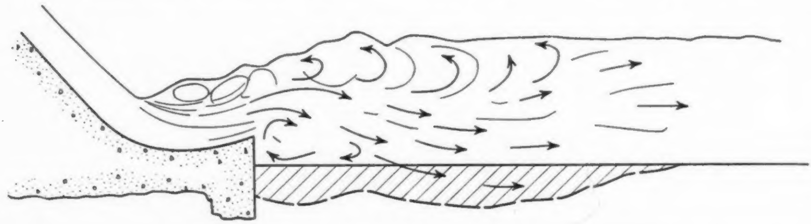


FIG. 19.—FLOW WITH SURFACE ROLLER

accompanied by a ground roller, which may not be of sufficient magnitude to produce serious scour conditions near the end of the apron, and in fact may even deposit material in that region as in the case of the Holtwood Dam with the river bed at El. 128. If the tailwater is raised sufficiently, the jet, rather than being deflected upward, is submerged and deflected downward, producing instead a surface roller, high velocities along the stream bed, and back eddies and turbulence at the end of the apron, as indicated by Fig. 19. The resulting scour is heavy and more general in nature, and energy dissipation occurs through diffusion of the jet into the tailwater, and excessive turbulence. The

latter type of flow is undesirable from the standpoint of erosion at the toe of the dam. At some point between these two tailwater elevations, the action will become unstable and the jet will alternately produce ground and surface rollers, the performance being first of the one type and then of the other. The unstable action is particularly undesirable because, in addition to the scouring effect of the water, loose bed material is swept alternately upstream and downstream with each change in action, exercising an abrasive effect on such bed material as might otherwise be undisturbed.

Because of this uncertainty in operation characteristics, the adoption of the Safe Harbor or Holtwood type of spillway apron in the design of other dams without careful and exhaustive check by model experiments might well lead to undesirable results, especially if the apron adopted is relatively short, with a comparatively flat angle to the horizontal. The need for intensive and careful study of this type of apron by hydraulic model experiments prior to its adoption as a final design for a proposed structure cannot be overemphasized.

The foregoing comments are in no way a reflection on the designs adopted for the Safe Harbor and Holtwood dams, and are presented only as a warning to engineers who might otherwise be tempted to adopt those designs for other structures without adequate analysis. The engineer should be cautioned to be absolutely certain that the test results are applicable to his problem; or, better yet, he should build a model and test it.

The paper by Mr. Davis is both interesting and informative. It summarizes the extensive tests made of the Safe Harbor and Holtwood developments and serves to emphasize the value that model studies have in the design of important hydraulic structures.

C. M. Allen,[5] M. Am. Soc. C. E.[5a]—There are at least two good reasons for doing the type of work described in this paper: (1) To make certain that the hydraulic structures in question will operate correctly in the field, thus giving the engineers confidence in the designs; and (2) to reduce the costs of these structures without interfering with the operation and efficiency. In one afternoon of laboratory work, in the writer's experience, the investigators were able to save $100,000 in concrete and excavation, and in addition they obtained better operating conditions. On the other hand, model tests have been made that called for considerable changes in design, and in some few cases the final cost was more than anticipated. However, it is always much more economical to make changes in the model than to correct mistakes in the field.

Mr. Davis is to be complimented on his performance of a fine piece of work.

---

[5] Prof. of Hydr. Eng., Worcester Polytechnic Inst., Worcester, Mass.

[5a] Received by the Secretary February 2, 1942.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

DESIGN AND CONSTRUCTION OF SAN GABRIEL DAM NO. 1

Discussion

BY MESSRS. FEDERICO BARONA, AND G. H. HICKOX

FEDERICO BARONA,[9] ESQ.[9a]—The information given in the paper by Mr. Baumann is interesting and constitutes a valuable contribution to the subject of the intelligent use of the materials available, in the construction of rock-fill and earth-fill dams.

The use of gravel or rock as coarse aggregate, bound by a sand-clay mortar, is undoubtedly an efficient manner of increasing the fill density and internal friction. If the coarse aggregate is added properly so that a certain limit is not exceeded, and so that sufficient compaction is assured, the composite fill can be made as watertight as the mixture of sand and clay mortar by itself. "El Palmito" Durango, in Mexico, is an earth-fill dam 280 ft high. In its construction, about 40% gravel was used as an addition to the sandy-clay loams available.

The addition of gravel has been economical since the depth of the layers of sand-clay in the borrow pits is limited by layers of gravel. By using the gravel, the borrow pits close to the site can be excavated to greater depths. Furthermore, the density of the fill has been increased.

The experience obtained in "El Palmito" Dam shows that the gravel-sand-clay mixture can be compacted more satisfactorily by means of smooth rollers. The gravel pieces act like the sheepsfoot elements of a roller to transfer the pressure from the smooth roller to the lower layers. For this kind of material, smooth rollers are more efficient than sheepsfoot rollers, because the latter tend to dislodge the pieces of gravel already embedded in the sand-clay mortar.

The trend in the construction of compacted fill dams must be toward the formation of soil concretes, with improvements in technique, grading, water content, etc., until they are as well known as cement concretes. Of course, if

NOTE.—This paper by Paul Baumann, M. Am. Soc. C. E., was published in September, 1941, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: November, 1941, by E. Soucek, Jun. Am. Soc. C. E.; December, 1941, by William P. Creager, M. Am. Soc. C. E.; January, 1942, by Joseph Jacobs, M. Am. Soc. C. E.; February, 1942, by Jacob Feld, M. Am. Soc. C. E.; and March, 1942, by Walter L. Huber, M. Am. Soc. C. E.

[9] Testing Engr., Mexican Commission of Irrigation, Mexico, D.F., Mexico.

[9a] Received by the Secretary February 2, 1942.

coarse aggregate is used in compacted fills, the friction to overcome is greater and so the compacting equipment must be much heavier.

Soil concretes have not been studied much, because generally, in the soil laboratory, only small specimens are tested, and only the fine portion (sand-clay mortar) can be included. San Gabriel Dam has the distinction of being the first project providing for the testing of larger specimens, with rocks included.

It is of interest to notice that in the beginning of Portland cement investigations, tests were made only on small specimens of neat cement or mortar. Later the technique of testing the entire mass of concrete was developed until 36-in. by 72-in. cylinders were tested.

In the case of soil concretes it would be worth while to determine permeability, settlement, compaction, and triaxial compression strength in specimens (24 in. diameter or more) sufficiently large to include gravel or rock such as is used in the fill.

Tests, in place, for permeability, settlement, compaction, etc., must be given serious consideration. They are especially important in the case of soil concretes as a means of approaching a greater knowledge of the true conditions that occur within the finished fill.

Experimental sections of fill compaction, as observed in San Gabriel Dam, are of great practical importance. In the Mexican Commission of Irrigation it is general practice to make these experimental sections whenever the construction of a new dam is begun. Thus it has been possible to determine many factors of technical and economic importance, such as: Thickness of layers to compact, number of trips for rollers, type and weight of equipment, etc. The measurement of settlement in embankments is also a general practice in the Mexican Commission of Irrigation.

Permeability tests, in place, have been made by the Commission to some extent, but for such an important soil characteristic the cooperation of all those engaged in the construction of earth dams is desirable to establish standard testing methods.

G. H. Hickox,[10] M. Am. Soc. C. E.[10a]—The part of Mr. Baumann's paper that describes the tests on the spillway model, particularly the tests on models at three different scales, is especially interesting. The results of these tests lead to Eq. 5. This equation can be developed if it is assumed that the velocity of flow through a horizontal element of area over the crest is

$$V = \sqrt{2\,g\,H^n} \qquad (16)$$

in which n is equal to 1.07.

If Eq. 5 with a constant coefficient is valid for the prototype as well as for the model, it leads to the curious relationship that the ratio of prototype discharge to model discharge varies as the 2.535 power of the length ratio instead of the 2.5 power predicted from considerations of dynamic similarity.

[10] Senior Hydr. Engr., TVA, Hydraulic Laboratory, Norris, Tenn.

[10a] Received by the Secretary February 26, 1942.

It has been supposed generally that discharge over a prototype spillway would be greater than that predicted from a model because of the relative reduction of friction loss in the prototype, and that the change in discharge would be represented by a change in the coefficient. The points of Fig. 24 seem to be equally well represented by three lines having a slope of 1.5 and different coefficients. It is unfortunate that data on the various models do not overlap.

It would be interesting to know if advantage was taken of the discharge illustrated in Fig. 33(*c*) to obtain a prototype verification of the curve of Fig. 16. Such verification of model tests is rare and is needed acutely.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

FUNDAMENTAL ASPECTS OF THE DEPRECIATION PROBLEM A SYMPOSIUM

Discussion

BY MESSRS. JAMES T. RYAN, JOHN C. PAGE, AND JOHN S. WORLEY

JAMES T. RYAN,[46] ASSOC. M. AM. SOC. C. E.[46a]—This Symposium reveals quite clearly both the extent and the sources of the confusion in current thought relative to the subject. The stresses to which language has been subjected in the processes of adaptation to the need of the machine age for agencies of description and expression have been at least as severe as those entailed by the machine upon metal, timber, and stone. By the same token, the genius that created the new industrial culture has had no harder task than those involved in defining the principles underlying it and explaining their meaning and scope. To do this it has been necessary virtually to create a new language in which a vast number of words and phrases in previous use have been adapted to purposes for which, in many instances, they were but poorly suited. The word "depreciate" and its derivatives, for example, originally applied solely to value, and dictionaries still define them as related to conditions that are measurable in terms of money.

With these definitions, of course, the writer has no contention. The engineer's trouble arises from an extension of the terms to conditions not contemplated in their original definition. With the growing importance of problems in physics, in chemistry, and in mechanics, arising from the decay of timber, the corrosion of metals, the disintegration of masonry, and the attrition of wearing surfaces, there came a need for a general term that would characterize all such processes in the same way that "depreciation" covered all influences that adversely affected values. Unfortunately, no new word seems to have

NOTE.—This Symposium was published in November, 1941, *Proceedings*. Discussion on this Symposium has appeared in *Proceedings*, as follows: November, 1941, by Messrs. Edwin F. Wendt and L. T. Fleming, and Anson Marston; December, 1941, by Messrs. William G. Atwood, George E. Goldthwaite, Nathan B. Jacobs, H. J. Flagg, Nelson Lee Smith, and C. Beverley Benson; February, 1942, by Messrs. Paul T. Norton, Jr., H. L. Ripley, and Carroll A. Farwell; and March, 1942, by Messrs. Thomas R. Agg, Conde B. McCullough, W. V. Burnell, Roger L. Morrison, Wallace B. Carr, W. L. Waters, A. G. Mott, David A. Kosh, E. G. Walker, and K. Lee Hyder.

[46] Cons. Valuation Engr., Pacific Gas & Elec. Co., San Francisco, Calif.

[46a] Received by the Secretary March 2, 1942.

been available, and the old one was appropriated for the purpose and has come into general use.

At the same time the accountants needed a word for a similar purpose. They were not interested in depreciation as the economists applied the term to declining values or as the engineers had begun to use it with reference to impaired physical integrity or functional efficiency. They were concerned with the problem of compensation for the consumption of capital resulting from use, regardless of how effected or indicated; and they too (probably also because no other term seemed to be available for the purpose) appropriated "depreciation" for the purpose, and gave to that word a new meaning consistent with the new uses to which they adapted it.

From these premises it seems pertinent to suggest that Professor Grant's "three fundamentally different depreciation concepts" may well be restated, perhaps as follows:

> First (and oldest), the economic concept, in which depreciation is related solely to loss in value, from any and all causes;
>
> Second, the engineering concept, in which it is essentially related to impaired physical integrity or functional efficiency, irrespective of their effects on values; and
>
> Third, the accounting concept, in which conclusions reached under the first and second concepts are applied to the measurement of capital lost or consumed in current operations.

Obviously, these concepts may lead to entirely different and unrelated conclusions.

Mr. Walls apparently prefers to limit consideration of the subject to the engineering concept, although he fully recognizes the engineer's responsibility to both accountant and economist. Professor Grant virtually ignores the engineer's concept—or, at least, subordinates it entirely to those of accountant and economist.

Messrs. Crum and Winfrey present an additional concept which, although primarily based upon economic considerations, is confused by adapting the definitions quoted by them toward an objectivity that seems to be related to political expediency rather than principle. It seems incredible that they should be able to deduce from their definition of depreciation (see heading "Value of Depreciation: Definition of Depreciation") the procedure for determining, from subsequent accounting records, the "true value" of a property outlined under the heading "Distinction Between Depreciation and Maintenance." The accountant cannot keep a record that will at any time exhibit such a result. His entries begin with a statement of cost. This cost may or may not be equivalent to value. The periodical entries recording the subsequent consumption of this original capital, either as depreciation, depletion, or amortization, may or may not be equivalent to concurrent "loss in value" or "decline in present worth" applicable to the same property. The accountant is not in a position to make such a determination and cannot assume responsibility for it. His records are kept primarily for an entirely different purpose, and their usefulness for that purpose would be impaired if not destroyed if he undertook to reflect in them current or periodical adjustments of invested capital to con-

form with "present true value." The accountant cannot assume this function of the economist; nor can the extent of physical or functional impairment, either as a periodical occurrence or as an accumulated total, be determined by him. That is the engineer's job. If he performs it properly and reports the results intelligently, the accountant will find his conclusions to be of inestimable value in maintaining the accuracy of accounting records, but neither of the two should be mistaken for the other.

It might be suggested that Professor Grant's admirable paper is faulty to the extent that it subordinates engineering analysis to economic considerations. His Examples 3 and 4 are calculations showing comparisons between the present values of specified annuities with different time factors, and are related to values rather than to a determination of depreciation as an engineering concept. Such calculations are of importance in many respects, but do not seem relevant to the questions under discussion, unless it is the intention of Professor Grant to suggest that the depreciation applicable to a given property may be measured properly by its earnings. This may have been one purpose of his Example 1, in which the accountant has a record of depreciation, and the engineer will doubtless discover convincing indications of it from an inspection of the structure, whereas the economist finds no evidence of loss in value and, therefore, no depreciation.

Under Messrs. Crum and Winfrey's suggested procedure, this sequence is reversed. If the accountant's records show charges against earnings for depreciation, a corresponding decrease in value is established, regardless of evidence to the contrary, or the subsequent discovery of error in the bookkeeper's estimates. The fact of value, although supported by conclusive evidence, is to be disregarded if inconsistent with accounting records. The obsession that a theory is so beautiful that it must be protected from injury by the distortion or suppression of contradictory facts is not new in engineering, literature, or law, but seems to be out of place here.

It should be emphasized that depreciation is not the only influence that limits the useful life or the earnings of structures or machinery, and that a decline in either usefulness or earnings does not progress uniformly with the passage of time. Any great industrial enterprise requires the coordination of various kinds of equipment to operate in each of its major processes. The depreciation of any given structure or machine, or any component element therein, may entail injurious effects upon other elements in the same structure or machine, or upon other structures or machines in the same plant or system. The replacement of one unit or element in such a plant or system frequently entails the necessity for the replacement of others that may be in excellent condition for further service. A change of detail in its products may involve extensive remodeling or discarding of plant equipment and facilities.

Physical condition is not, therefore, a safe basis for a forecast of future usefulness; nor is an estimate of life expectancy a satisfactory basis for determining physical condition. The suggestion, so frequently implied in this Symposium, that an accurate determination of depreciation may be arrived at by reference to the calendar (for example, from estimates of probable life expectancies) is fundamentally unsound.

The engineer's perspective of depreciation embraces both causes and effects, and involves the consideration of measures calculated to abort causative influences and to minimize or avert resultant damages. Accountants and economists alike have profited by making full use of his judgment and experience.

John C. Page,[47] M. Am. Soc. C. E.[47a]—In analyzing "Fundamental Aspects of the Depreciation Problem: Relationship to Public Works and Government Finance," Messrs. Crum and Winfrey have brought forth for inspection some of the differences between governmental and private accounting procedures. Emphasizing the lack of consideration of depreciation in governmental administration, the authors suggest that depreciation should receive a proper place in this field if accounting of public funds is to be other than a partial truth.

Federal officials charged with the administration of public works must observe, of course, the laws of Congress and the rules promulgated by regulatory bodies deriving their authority from the Congress. These laws and rules may be sufficiently restrictive to prevent the accumulation of reserve funds for replacements. This is not invariably the case, however, and, even in cases where maintenance of a reserve account is not practicable, recognition can be given to depreciation of property as a factor to be considered in planning future operations. The Bureau of Reclamation, for example, is engaged in the construction of large engineering projects, the operation and maintenance of major power plants, and the operation and maintenance of extensive irrigation systems. In each of these three primary activities, the principles of depreciation are recognized in the attendant accounting procedures. As an illustration of the applicability of these principles, and in the hope that something of value may be contributed to the discussion, a résumé of the accounting system of the Bureau of Reclamation is submitted.

The Bureau of Reclamation of the U. S. Department of the Interior since 1902 has been engaged in the construction, operation, and maintenance of engineering works designed to reclaim for agriculture the arid and semi-arid valleys of the Western States, to produce power, and to provide flood control and related benefits. Varying in size from the large-scale Grand Coulee Dam (Washington), Boulder Dam (Arizona-Nevada), and Shasta Dam (California) to the relatively small Great Plains projects, the Bureau has had the responsibility for expending more than $690,000,000 on engineering construction alone since 1902. The accurate accounting of this large sum has been in accordance with standardized regulations.

Before continuing the discussion of the subject, a general outline of the Bureau's financial structure may be helpful. The federal reclamation of arid lands in the West, concerned with the conservation of water, the production of electrical energy, and the created potentialities for the establishment of independent farm homes, is essentially a business enterprise. The capital is provided by a revolving fund known as the Reclamation Fund, which is supplemented by occasional special appropriations. The money expended is con-

---

[47] Commr., Bureau of Reclamation, U. S. Dept. of the Interior, Washington, D. C.

[47a] Received by the Secretary March 3, 1942.

tracted for repayment by the users of the finished product. The cash collected by the government as reimbursement for construction of one project becomes capital for the construction of another project.

The repayment plan or amortization of the construction cost of the completed project is based on an exhaustive study of crop production, income from crop yields, and use of electrical energy. The total repayable obligation must be within the ability of the consumer to pay, and the repayment must be completed within maximum periods fixed by statute. At the end of the amortization period the project becomes in effect the property of the users, although ratification of the transfer by Congress is necessary. The anticipated costs after construction are routine operation and maintenance costs and occasional major replacements, which are made by the users' organization or by the Bureau acting as their representative.

In so far as its power developments are concerned, the Bureau of Reclamation recently has adopted the Federal Power Commission's uniform system of accounts for public utilities and licensees where practicable. The Boulder Canyon Project Adjustment Act of July 19, 1940, definitely provided for a rate structure that included, in addition to operation and maintenance, amortization at 3% interest of moneys advanced by the Treasury of the United States for financing its construction and the establishment of an adequate reserve for replacements. The principle of depreciation is also recognized in the establishment of rates for the sale of electrical energy from its other power developments and in the establishment of rates for the sale of water from its irrigation facilities for purposes other than irrigation.

Depreciation as used by the Bureau of Reclamation in its power-plant operations is considered to be the reduction over an estimated period of time of the original cost or value of a structure, facility, or unit of equipment, due to its deterioration by wear or tear, or otherwise. It is so recognized for the purpose of insuring accurate and proper fund accounting.

The Bureau further has recognized the advisability of depreciation accounting for its irrigation operations; however, existing law and contractual commitments do not permit charging depreciation, as such, as an item of cost of operation because annual fixed charges are collected from irrigation beneficiaries for the amortization of the fixed capital indebtedness, in addition to operation and maintenance charges for irrigation service.

In order that the consumer shall repay only the actual cost of construction to the government, the cost-accounting system used during the construction period allocates all costs to the principal and physical features of the project. This system employs a commercial practice of depreciation for all nonexpendable property during construction, allowing a credit to the project costs for the salvage value of unused plant equipment and supplies on completion of construction. The records accumulated in this fashion, in addition to determining the exact sum that must be repaid by the water users or power consumers, have served to provide a reliable basis for estimating the costs of new projects or parts of a project, or the cost of replacement of structures on operating (completed) projects.

Costs during construction are maintained by principal features, of which a single reclamation project usually contains two or more. Principal features represent a general division of the project into its primary parts or units; the nature or function of each feature is usually implied by its name, such as examination and surveys, storage works, canal systems, etc. Principal features are further broken down into physical features. A breakdown of principal features into physical features is made for the purpose of collecting exact costs on separate and distinct jobs or units or divisions of a project. They represent some tangible section of a project when construction is completed, such as a group of water-control structures, a tunnel, etc.

Supervision, labor, and materials comprise the bulk of construction costs. Obviously, an accurate distribution of these costs to units can be made with no unusual difficulty. However, many items of equipment are not fully "deteriorized" in the construction of a single unit and may be used in a number of units under varying circumstances. Such equipment is termed "nonexpendable equipment," and the method of distributing its cost to individual units is through charges based on depreciation of the item.

The costs of equipment and its erection into a working unit are accumulated into what is known as a "plant account." The costs of the equipment are absorbed into the cost of features upon which they are used by the monthly transfer from the plant account of an amount that is proportioned to the work accomplished during the month. The transfer of these costs involves two elements: (1) "Plant" charges proper, which cover the direct costs of installation and erection; and (2) equipment charges, which cover the depreciation to be charged to the feature for the use of nonexpendable equipment. The transfer of plant charges proper presents no unusual problem. The transfer of equipment charges through depreciation is described in the following paragraphs.

Based on the Bureau's experience, different rates of depreciation for the various kinds and classes of equipment are established. The rates for motor vehicles are based on a computation of percentage of original cost against units of mileage. For instance, motor-truck costs are usually charged off on a basis of 40,000 miles total travel; the charge for the first 10,000 miles is estimated at 40% of the original cost, for the second 10,000 miles at 30%, for the third 10,000 miles at 20%, and for the fourth 10,000 miles at 10%. The reason for this graduated scale is to decrease the depreciation charge as the operation cost increases. The mileage basis of rates varies, depending on the nature of the work to be performed by the vehicle, but the foregoing rates are generally used for the Bureau's field operations.

Small tools are charged out on a monthly rate to the plant or work on which they are used. Charges usually are made to each feature based on a percentage of the value of tools in use on the feature at a rate of 8% monthly or they are based on a charge that is a percentage of the monthly cost of the feature, whichever is most appropriate. Depreciation on small tools varies greatly in the different classes of work and, in case the percentage selected is found to be either too high or too low, it is changed to agree as nearly as practicable with actual conditions.

Other articles of major equipment are also grouped into "plants" upon which monthly depreciation charges are made as follows:

Name of plant	Percentage rate per month
Camp construction plant equipment	5
Construction plant equipment	4
Corral plant equipment	4
Engineering, field plant equipment	2
Hospital plant equipment	3
Mercantile store plant equipment	3
Miscellaneous equipment and tools	8
Office plant equipment	2
Preliminary investigations and testing plant equipment	4
Railroad plant equipment	3
Sawmill plant equipment	5
Shop plant equipment	3
Storehouse plant equipment	3
Telephone system (temporary plant) equipment	3

These charges are made on appraised value, determined annually, and not on original cost, except in the case of the first year for new equipment. The charge is made for the entire period the equipment is retained for use on the feature to which it is charged, regardless of whether or not its use is continuous during that period, except that no charge will be made when the performance of the class of work on which the equipment is used is suspended for one month or more. If the equipment is used more than an average of one shift per day, the charge is increased accordingly. No charge is accrued on equipment in inventory, if properly housed and protected.

The plant accounts are charged with all extensive equipment repairs in order to insure distribution of cost over a longer period than the month in which extensive repairs are made, thus arriving at a better unit cost of work in progress. The cost of such repairs is distributed in the same manner, and at the same rates per month, as the charges for articles of equipment to which the repairs are made. Minor or ordinary maintenance repairs, however, are charged directly to the work.

Under the Bureau's procedure for depreciating nonexpendable property during construction of a project, many years might be required to depreciate each item fully—certainly many years more than it takes to construct a project. Some provision must be made for disposition of the balances for undepreciated values that remain on the books at the close of the construction period. Upon completion of the work for which equipment or plant has been used (and this may occur at any time during the construction period, not necessarily at the completion of construction), a salvage value is fixed by a board of survey. The equipment is then transferred to another project, or sold, using the appraised value. The appraisal is based on the actual value as represented by the physical condition and the current market value of similar equipment.

Adjustments are necessary to reconcile differences between the appraised values and the undepreciated balances.

The procedure just outlined applies to the construction activities of the Bureau of Reclamation. It enables the Bureau to determine precisely the cost of the entire project and all of its major subdivisions. The costs repaid by the users are no more or no less than are required by the governing statutes.

As previously mentioned, the Bureau is in the unique position of not only constructing projects, but frequently remaining as the operating agency following completion of the works. The handling of depreciation on power projects is controlled by rules of fixed procedure that are too complex in character to explain in this discussion. Essentially, it is that a certain portion of power revenues are set aside each year in a reserve fund for replacement of the major items of the plant. The accumulation of these reserves has a direct relation to the estimated service lives of the items of the plant.

The depreciation problems in connection with operating and maintaining completed irrigation projects (as distinguished from power projects) are similar to those encountered in any operating enterprise. Operation and maintenance funds for reclamation projects are either advanced by the water users at the beginning of each year or are appropriated by Congress and repaid by the water users at the end of each year. It would be desirable to set aside in a reserve account, each year, from these appropriations or advances, a sum that could be deposited or invested in order that at the end of a fixed period of years adequate capital would have been amassed to provide for replacement of major structures with a limited service life. As a matter of policy, the Bureau favors the establishment of such reserve accounts. Their establishment requires the approval of the water users, and it is expected that the maintenance of larger reserve accounts will become more widespread as their value is more fully appreciated. Whether additional legislation would be helpful in providing an answer to the depreciation problem for federal public works cannot be known without further study, but certainly this Symposium has been helpful in revealing the weaknesses of the subject and the need for uniformity in handling it.

JOHN S. WORLEY,[48] M. AM. SOC. C. E.[48a]—The subject of this Symposium divides itself into two categories—one, the nature of depreciation; the other, the treatment to be given to it in the various situations having to do with private and public businesses and public utilities. In ascertaining the nature of depreciation one deals with a specific and tangible fact as of some instant of time, whereas the matter of its treatment is solely one of public or private policy, which, in some cases, must be made effective by legislative action.

Every business property, when acquired by purchase or construction, is for the purpose of providing service. That is, during a period of time, with normal maintenance, it is capable of manufacturing a certain quantity of products, thousands of barrels of flour, gross of shoes, kilowatts of electricity, ton-miles of goods-carrying, locomotive miles, car miles, passenger miles, cubic feet of gas, and all other products of industries. As a plant is operated these

[48] Prof. of Transportation Eng., Univ. of Michigan, Ann Arbor, Mich.

[48a] Received by the Secretary March 6, 1942.

units of service are consumed, but as maintenance, repairs, and renewals are made these units are in part replenished. It seems logical and reasonable to define depreciation as the loss of service units.

There are many types of property, the normal maintenance, repairs, and renewals of which will keep it in first-class operating condition. When a property reaches this stage the annual maintenance, repairs, and renewals just equal the average losses and there is no further accumulation of depreciation. This condition is found in the tie system of the American railroad which is receiving its regular maintenance, repairs, and renewals. Where the annual use is fairly uniform and the maintenance, repairs, and renewals become uniform, the annual losses are "made good" currently and it may be said that there has been no depreciation for that year. When the losses are due to wear and tear from use, the annual depreciation is in a direct proportion to the use. When the losses are from deterioration due to weather the depreciation is uniform from year to year.

Obsolescence, as of a specific time, or during a certain year, is more difficult to determine than any other kind of depreciation. Where there is a uniform change in an art from year to year, and where, at the end of a certain period, a machine is found to be obsolete and inadequate, it would seem reasonable to say the obsolescence had occurred in a uniform manner each year. On the other hand, where a machine becomes obsolete and inadequate because of an invention, it would seem that the major part of the obsolescence had occurred in the year of the invention and its adoption. However, it can be reasoned that from past experience these new inventions are to be expected and therefore obsolescence does occur annually.

A splendid example of the irregularity with which units of service may be used up is found in the heavy artillery of the United States Army. Through long experience it has been found that each piece of artillery, depending upon its type, can be fired a certain number of rounds, after which it is no longer accurate in its range nor safe from explosion. As a result, each piece is fired only the determined number of rounds before it is retired from service. Now if a new gun is held in reserve for two years, and one half of the units of service are fired within one week, there has been no depreciation for the two years, and 50% of its value is depreciated within one week. However, if the gun had been fired once a day the depreciation would accumulate from day to day until all the units of service had been used up, at which time the gun would have been retired.

As a rule, the engineer is called upon to determine depreciation as a fact, as of a particular date. At no time should the engineer lose sight of the fact that depreciation cannot be effected by subtle metaphysical reasoning or by any other process. Whenever the engineer deals with this subject the quality of his work will be in direct proportion to the degree with which he ascertains and records the facts. The work of the accountant varies from that of the engineer in that his estimates must be made in advance of the fact. He estimates what the probable depreciation will be so that he may make proper debits against annual income.

With the knowledge of the nature of depreciation, a policy with reference to its treatment can be adopted intelligently. In the administration of a policy either an engineer or an accountant, or both, are the parties usually employed to participate in implementing the policy adopted. This requires the determination of the depreciation in certain units, which is translated into the medium of exchange. This work is one of professional abilities and skills and there should be little or no difference in the results determined by either engineers or accountants, and such differences as may exist should be readily reconciled.

It is the writer's belief that much of the difficulty which has been experienced in dealing with depreciation has been the attempt to have it controlled by a predetermined definition or thought of some social school or economic order. He also believes that the solution of this problem, like many other social problems, should be predicated upon as complete a knowledge of the nature of the subject as possible, from which a policy of treatment is determined.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

DESIGN OF ST. GEORGES TIED ARCH SPAN

Discussion

BY JACOB KAROL, ESQ.

JACOB KAROL,[5] ESQ.[5a]—An interesting description of the design of a tied arch span is contained in this paper. This type of structure contains approximately the same amount of metal as a simple truss span carrying the same loads, and is certainly more pleasing in appearance than any type of through truss. Hence, it should find increasing use in locations where esthetic considerations are of some importance.

The writer wishes to indicate certain simplifications in the preliminary design procedure. If the integration is performed in Eq. 8*b*, the expression for the influence line for *H* becomes

$$H = \frac{5\,l}{8\,h}\,(k^4 - 2\,k^3 + k) \qquad (19)$$

in which k is the proportionate distance from the left end of the span to the unit load. Table 1 shows the comparison between the values of H from this formula and the final values of H given by the author.

TABLE 1.—COMPARISON OF VALUES OF *H*

Load at panel point	By formula	Author
1	0.240	0.2358
2	0.468	0.4595
3	0.674	0.6625
4	0.850	0.8372
5	0.987	0.9772
6	1.080	1.0772
7	1.128	1.1300

Thus, it is seen that a very close approximation to the final influence line for H, and hence to the final direct stresses, can be obtained if only the span length and the center rise are known. Having the influence line for H, influence lines for M may be determined from Eq. 9. It should be assumed, however, that the live-load moments determined from these influence lines represent the total moments resisted by the arch rib and tie girder.

NOTE.—This paper by J. M. Garrelts, Assoc. M. Am. Soc. C. E., was published in December, 1941, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: February, 1942, by R. W. Abbett, M. Am. Soc. C. E.

[5] Howard, Needles, Tammen & Bergendoff, Kansas City, Mo.

[5a] Received by the Secretary March 5, 1942.

The next step is to determine the division of the total moment between the girder and the rib. The author's procedure, using Eq. 16, requires that the moments of inertia of the rib and the girder be known in advance. However, this division can be obtained approximately if only the depths of the rib and the girder are known. From the flitched-beam concept, the angle changes along the rib are equal to those along the girder (except for the change in length of the hangers, which effect will be neglected). Hence,

$$\frac{M_G\,dx}{E\,I_G} = \frac{M_R\,ds}{E\,I_R} = \frac{M_R\,dx}{E\,I_R\cos\alpha} \qquad (20)$$

from which

$$\frac{M_R}{M_G} = \frac{I_R\cos\alpha}{I_G} \qquad (21)$$

If the sections of the rib and girder are assumed to be proportional to their direct stresses, then

$$A_G = C_1\,A_R\cos\alpha \qquad (22)$$

in which C_1 = the ratio of the unit stresses due to the direct loads in the rib and girder, respectively, $= \dfrac{f_{R(d)}}{f_{G(d)}}$.

The value of C_1 may be obtained as follows: Assume a value of the fiber stress due to bending in the girder. In the structure under consideration this value is about 8,000 lb per sq in. on the gross section. This value may be determined approximately at a particular section, using the direct loads and moments previously found in the following equations:

$$f_G = \frac{H}{A_G}\left(1 + \frac{2.5\,e}{d_G}\right) = f_{G(d)}\left(1 + \frac{2.5\,e}{d_G}\right) \qquad (23)$$

in which f_G = allowable unit stress in girder on gross section; H = tension due to dead load and live load; $e = \dfrac{M}{H}$; and M is the moment due to dead load and live load.

Since $f_G = f_{G(b)} + f_{G(d)}$, by substitution in Eq. 23,

$$f_{G(d)} = \frac{f_G\,d_G}{d_G + 2.5\,e} \qquad (24a)$$

and

$$f_{G(b)} = \frac{f_G \times 2.5\,e}{d_G + 2.5\,e} \qquad (24b)$$

For the case presented in the paper, assume that the allowable stress in the girder $f_G = 80\% \times 24{,}000 = 19{,}200$ lb per sq in. on the gross section; also, for the allowable stress in the rib, $f_R = 19{,}400$ lb per sq in. Since the unit stresses in the girder and rib due to bending are approximately proportional to their respective depths,

$$f_{R(b)} = f_{G(b)} \times \frac{d_R}{d_G} \qquad (25)$$

Substituting numerical values, $d_R = 40$ in. and $d_G = 108$ in., and $f_{R(b)} = 8{,}000 \times \frac{40}{108} = 3{,}000$ lb per sq in. Hence, the unit stresses due to direct loads are $f_{G(d)} = 19{,}200 - 8{,}000 = 11{,}200$ lb per sq in. in the girder and $f_{R(d)} = 19{,}400 - 3{,}000 = 16{,}400$ lb per sq in. in the rib, from which $C_1 = \frac{16{,}400}{11{,}200} = 1.46$. Substituting in Eq. 22, $A_G = 1.46\, A_R \cos \alpha$.

The moments of inertia of the girder and rib may be written

$$I_G = C_2\, A_G\, d^2_G \qquad (26a)$$

and

$$I_R = C_3\, A_R\, d^2_R \qquad (26b)$$

Since the girder and rib are both box sections, assume $C_2 = C_3$ for simplicity. Then

$$\frac{I_R}{I_G} = \frac{C_3\, A_R\, d^2_R}{1.46\, C_2\, A_R\, d^2_G \cos \alpha} = \frac{d^2_R}{1.46\, d^2_G \cos \alpha} \qquad (27a)$$

and

$$\frac{M_R}{M_G} = \frac{d^2_R \cos \alpha}{1.46\, d^2_G \cos \alpha} = 0.685\, \frac{d^2_R}{d^2_G} \qquad (27b)$$

Substituting numerical values, $\frac{M_R}{M_G} = 0.685 \times \frac{(40)^2}{(108)^2} = \frac{1}{10.6}$; $\frac{M_R}{M_T} = \frac{1}{10.6 + 1} = \frac{1}{11.6} = 8.6\%$; and $\frac{M_G}{M_T} = \frac{10.6}{11.6} = 91.4\%$.

This agrees very well with the distribution indicated by the author. It may be stated that, by assuming $C_1 = 1$, without making any calculations whatever to determine its value, the distribution of the total moment to the rib and the girder is 12.1% and 87.9%, respectively.

Hence, the foregoing procedure results in a preliminary determination of the division of the total moment between the rib and the girder, and of the direct stresses in the rib and the girder, with sufficient accuracy that the sections determined therefrom should be very near the final sections. It is worth emphasizing that these results can be obtained when only the span length, the center rise, and the depths of the rib and girder are known. Having thus obtained sections on which to base an analysis, the final analysis would follow the author's procedure, using Eqs. 16, 17, and 18.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

DEVELOPMENT OF TRANSPORTATION IN THE UNITED STATES

Discussion

BY MESSRS. J. L. CAMPBELL, W. W. CROSBY, AND W. B. IRWIN

J. L. CAMPBELL,[15] M. AM. SOC. C. E.[15a]—The paper presented is a valuable historical review of the development of transportation by railways, waterways, highways, airways, and pipe lines in Continental United States.

Between the several kinds of transportation, important issues lie in matters of need and capacity, extensions or enlargements, costs, competition, regulation, and coordination of transportation services. Out of right settlements or adjustments of such issues, unified, adequate, and cooperative transportation would flow at fair costs promoting national prosperity and welfare.

Developments of transportation on the highways and the airways are in a fluid status making probable no early sound and adequate solution of the entire problem which should be assigned to a commission composed of men of best ability and experience for the task. Such solution should be translated into appropriate legislation. Meanwhile, "first aid" is necessary for the preservation of railway transportation, the major part of all transportation. Such aid lies within the discretion of the Interstate Commerce Commission.

In terms of volume, value, and indispensability, railway transportation towers above its competitors. It is the backbone of transportation to which other forms should be adjusted in equitable relations and fair competition so that the backbone shall not be broken or incapacitated in the major service it must render. As between railway transportation on one side and waterway and highway transportation on the other side, regulation and competition have been unfairly adverse to transportation by railway because transportation by waterway and highway has not paid the full cost of its service and has not been subjected to the rigors of railway regulation. Regulation should cover fairly all competitive transportation.

NOTE.—This paper by J. E. Teal, M. Am. Soc. C. E., was published in December, 1941, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: February, 1942, by Messrs. Fred Lavis, and William J. Wilgus; and March, 1942, by Joseph B. Eastman, Esq.

[15] Chf. Engr., Northwestern Pac. R. R. (*Retired*), Oakland, Calif.

[15a] Received by the Secretary January 26, 1942.

While the trend of railway revenue rates has been generally downward or *in status quo*, the trend of unit costs of labor and materials used by the railways has been upward. Since 1920, there has been growing diversion of traffic from the railways to the highways. This is reason enough for the present financial condition of the railways wherein about one third of railway mileage is now in receivership and the remainder of it is unable to earn a fair return upon investment except in periods of abnormal traffic of great volume. For normal traffic, railway revenue is generally below railway expense. This result is largely due to too much limitation of railway revenue by fiat of law and dicta of the Interstate Commerce Commission. A railway dividend is now almost a curiosity to be placed under a glass case in the Smithsonian Institution.

In volume, excellence, and relatively low cost of transportation, the American railway system is unequaled anywhere. It has avoided complete shipwreck and has continued operation as a private enterprise under rigorous regulation by improvements and efficiencies in its plant and service which have reduced cost of transportation and partly bridged the gap between revenues and expenses. It is probable that a point would be reached beyond which the cost of additional improvement and efficiency would widen rather than close the gap. It is improbable that inadequacy of revenue can be completely and permanently eliminated so long as revenue rates are frozen and unresponsive to economic conditions. Unless the rates are substantially maintained in such relation to cost of service as will assure a fair return to private initiative and enterprise, the inevitable end will be government ownership and operation. Thereby, the record indicates that cost of transportation would increase.

Commercial traffic on the highways being so directly and highly competitive with railway traffic should be closely coordinated with its competitor and be subject to the laws controlling railway transportation to the end that regulation of railways and highways as commercial transportation agencies shall be common and equitable to both. Multiplicity of transportation agencies on the highways retards rather than expedites traffic thereon. In so far as the public interest and convenience are promoted thereby, commercial highway traffic should be conducted by the railways.

The national system of highways came into being by the simple and relatively inexpensive process of covering the surface of existing wagon roads with some sort of pavement, the alinement of the roads having remarkable similarity to the meanderings of the covered wagons of the heroic days of Westward Ho. The result is a system of paved highways which, in alinement, gradients, widths, and paved surfaces, is distressingly and dangerously inadequate for the volume and speed of today's highway traffic. Only a beginning has been made on necessary nation-wide reconstruction of this system to standards indispensable to highway requirements of today and the future. This is a reconstruction job which, when done, will have cost vastly more than the original paved roads.

Transportation on the Great Lakes being comparable to transportation on the oceans is the most economical transportation within Continental United

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

DEVELOPMENT OF TRANSPORTATION IN THE UNITED STATES

Discussion

BY MESSRS. J. L. CAMPBELL, W. W. CROSBY, AND W. B. IRWIN

J. L. CAMPBELL,[15] M. AM. SOC. C. E.[15a]—The paper presented is a valuable historical review of the development of transportation by railways, waterways, highways, airways, and pipe lines in Continental United States.

Between the several kinds of transportation, important issues lie in matters of need and capacity, extensions or enlargements, costs, competition, regulation, and coordination of transportation services. Out of right settlements or adjustments of such issues, unified, adequate, and cooperative transportation would flow at fair costs promoting national prosperity and welfare.

Developments of transportation on the highways and the airways are in a fluid status making probable no early sound and adequate solution of the entire problem which should be assigned to a commission composed of men of best ability and experience for the task. Such solution should be translated into appropriate legislation. Meanwhile, "first aid" is necessary for the preservation of railway transportation, the major part of all transportation. Such aid lies within the discretion of the Interstate Commerce Commission.

In terms of volume, value, and indispensability, railway transportation towers above its competitors. It is the backbone of transportation to which other forms should be adjusted in equitable relations and fair competition so that the backbone shall not be broken or incapacitated in the major service it must render. As between railway transportation on one side and waterway and highway transportation on the other side, regulation and competition have been unfairly adverse to transportation by railway because transportation by waterway and highway has not paid the full cost of its service and has not been subjected to the rigors of railway regulation. Regulation should cover fairly all competitive transportation.

NOTE.—This paper by J. E. Teal, M. Am. Soc. C. E., was published in December, 1941, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: February, 1942, by Messrs. Fred Lavis, and William J. Wilgus; and March, 1942, by Joseph B. Eastman, Esq.

[15] Chf. Engr., Northwestern Pac. R. R. (*Retired*), Oakland, Calif.

[15a] Received by the Secretary January 26, 1942.

While the trend of railway revenue rates has been generally downward or *in status quo*, the trend of unit costs of labor and materials used by the railways has been upward. Since 1920, there has been growing diversion of traffic from the railways to the highways. This is reason enough for the present financial condition of the railways wherein about one third of railway mileage is now in receivership and the remainder of it is unable to earn a fair return upon investment except in periods of abnormal traffic of great volume. For normal traffic, railway revenue is generally below railway expense. This result is largely due to too much limitation of railway revenue by fiat of law and dicta of the Interstate Commerce Commission. A railway dividend is now almost a curiosity to be placed under a glass case in the Smithsonian Institution.

In volume, excellence, and relatively low cost of transportation, the American railway system is unequaled anywhere. It has avoided complete shipwreck and has continued operation as a private enterprise under rigorous regulation by improvements and efficiencies in its plant and service which have reduced cost of transportation and partly bridged the gap between revenues and expenses. It is probable that a point would be reached beyond which the cost of additional improvement and efficiency would widen rather than close the gap. It is improbable that inadequacy of revenue can be completely and permanently eliminated so long as revenue rates are frozen and unresponsive to economic conditions. Unless the rates are substantially maintained in such relation to cost of service as will assure a fair return to private initiative and enterprise, the inevitable end will be government ownership and operation. Thereby, the record indicates that cost of transportation would increase.

Commercial traffic on the highways being so directly and highly competitive with railway traffic should be closely coordinated with its competitor and be subject to the laws controlling railway transportation to the end that regulation of railways and highways as commercial transportation agencies shall be common and equitable to both. Multiplicity of transportation agencies on the highways retards rather than expedites traffic thereon. In so far as the public interest and convenience are promoted thereby, commercial highway traffic should be conducted by the railways.

The national system of highways came into being by the simple and relatively inexpensive process of covering the surface of existing wagon roads with some sort of pavement, the alinement of the roads having remarkable similarity to the meanderings of the covered wagons of the heroic days of Westward Ho. The result is a system of paved highways which, in alinement, gradients, widths, and paved surfaces, is distressingly and dangerously inadequate for the volume and speed of today's highway traffic. Only a beginning has been made on necessary nation-wide reconstruction of this system to standards indispensable to highway requirements of today and the future. This is a reconstruction job which, when done, will have cost vastly more than the original paved roads.

Transportation on the Great Lakes being comparable to transportation on the oceans is the most economical transportation within Continental United

States. This is not true of the rivers and canals whereon ton-mile cost is more than on the railways. Relative to the railways, rivers and canals are few in number, limited in mileage, non-elastic in location, and incapable of serving more than a small part of the total tonnage of the country. The railways have surplus capacity more than sufficient to carry the tonnage carried on the rivers and canals. All transportation requirements could be served without the rivers and canals. That would be impossible without the railways. In general, vast expenditures for navigation on rivers and canals have been made without economic justification. Such expenditures should cease.

A vital function of river channels is to carry away the flood waters of the continent. The drainage capacity of these channels should not be diminished by navigation structures without weighty reasons therefor. This was brought vividly to mind during the great 1937 flood on the Ohio. There is some conflict between flood control and navigation on rivers.

Waterway transportation should be required to raise its service rates sufficiently to cover total cost of plant and service and yield a fair return on investment so that its competition shall be fair to its competitors who must fix their rates as described. This also applies to airways and pipe lines.

The airways are too young for final conclusions thereon.

W. W. Crosby,[16] M. Am. Soc. C. E.[16a]—In this paper the author has set forth, most interestingly, many of the facts regarding the appearance and growth of the various facilities. He states that under present conditions all are competitive and that "unfair competition" has resulted. From purely financial viewpoints many observers would have the same view; but that is not the only aspect that should be considered in attempting corrections of the existing situation.

The development of waterways had a basis of regional interest besides the economic one. The railways had a considerable basis of national interest underlying their economic one—regardless of their prostitution by "high finance." Pipe lines were based almost wholly on financial interest, although a secondary result may have been to advance regional interests. Air transport has developed from a national, if not international, interest. The interest underlying highway development has been almost wholly local or parochial.

Considering these underlying motivations, it becomes apparent that, although some of the much-needed coordination could be had beneficially by cooperation (and possibly by some regulatory action), a more fundamental remedy must be sought. Legislation against the developments of applied science is not the American way to protect the Nation; nor is it even in the financial interest. Cleaner, clearer, and broader thinking than has prevailed is needed for the national welfare as well as for National Defense.

All through the warp and woof of highway development have run such threads as "taking the farmer out of the mud," "all-year roads," "farm-to-market highways," "express ways for faster traffic," and "safer and more

[16] Cons. Engr., Coronado, Calif.

[16a] Received by the Secretary March 16, 1942.

commodious highways." A very few suggestions of national interest occur. One was indicated perhaps by Congressional provisions for the national roads mentioned by the author. Possibly a second short thread may be found in the "National Highway" movement and others early in the twentieth century. Most of them broke off rather shortly.

The writer believes that careful analysis proves that even these latter were more regional at the bottom than national. The combining of local and regional highway systems to form "national highways" has been more fortuitous than otherwise, and it occurred as an afterthought rather than as an original concept for national benefit. L. I. Hewes,[17] M. Am. Soc. C. E., has stated that "The history of this strategic net [of national or defense highways] goes back to the Federal Highway Act of 1921"; but the earlier results were simply the linking up, for long-distance facility, of existing roads into a "PershingMap" in 1922. A revision of the Pershing Map was made in 1939, and was thereafter called the "Highway Map Showing Main Traffic Routes of Military Importance." Another revision was made under date of May 15, 1941.

Then Mr. Hewes states that "These main routes are intended to connect all important foci of actual or potential importance for defense. Clearly, the strategic net is used much more by civil than by military traffic." The foregoing quotations seem to the writer to support his statement preceding them.

With the possible exception of the "Harrisburg-Pittsburgh Turnpike," the writer remembers no highway in the United States that has been originated from a conception of the national, or even of a major regional, interest. Italy and Germany did do this many years ago; but the results so far obtained in the Unites States have come merely from piecing together local roads built for local ends and from narrow conceptions of what would gratify local or small regional desires. The federal influence has not been exerted effectively otherwise, and even as late as 1941 federal agencies seem to have continuously disapproved the broader idea.

A further fact in explanation of the insufficiency of present improved highways in National Defense is that the federal road authorities have clung too long to their announced principle that "the improvement of a highway is solely justified by the savings in transportation costs." Another fact may be that state highway authorities too often have been concerned mainly with the influence of highway expenditures on votes. The writer already has emphasized[18] that this criticism is not directed against engineers as such. At times, however, they are forced to share it.

The author states (see heading "Investment and Traffic") that the causes of "The changes in the transportation 'picture'" have been "in the broad sense, * * * entirely economic"—that is, "satisfactory service at the least direct expense." Does he not go too far? The writer thinks that Mr. Eastman (quoted by the author) has showed[19] a more complete understanding when he stated "Competition between different forms of transportation * * * has

[17] "Better Highways for National Defense," by L. I. Hewes, *Civil Engineering*, January, 1942, p. 10 *et seq.*

[18] *Transactions*, Am. Soc. C. E., Vol. 102 (1937), p. 1115.

[19] "The Transportation of Tomorrow," by Joseph B. Eastman, *Civil Engineering*, January, 1942, p. 1.

become essentially a warfare * * *" that "compels attention to the needs and desires, and even the foibles, of the public which is being served."

As the writer has contended heretofore, too great reliance has been placed in highway matters on mathematical formulas, financial returns (real or figurable), local political influences, and personal ambitions, with not enough reasonable regard for convenience, safety, and national interests. He does not believe that the durable solution of the desired coordination will be found in "transportation costs" alone. If "convenience" is to be considered, it may even be that highway competition with railway facilities will be increased. "Technocracy" has its appeal to the scientific mind; but as such its further reasoning was undermined long ago by the truths of Kipling's "Imperial Rescript."

Coordination is needed today. It is even indispensable. The author mentions this but seems to believe that the "coordination" desired is the subordination of other forms of transportation to the financial benefit of the railways. Neither the history of the railroads nor the developments of transportation science justify such a conclusion, economically or otherwise. The Transportation Act that he quotes seems to the writer to indicate a much broader view; and Mr. Eastman himself seems on the side of the writer.

Recently the writer had the occasion to mention to the National Resources Planning Board an instance of deplorable lack of coordination under his personal observation. The reply, dated September 19, 1941, was that "unfortunately, co-ordinating machinery for an over-all transportation approach (to the local problem) does not now exist in the Federal Government." To the writer's further suggestion that possibly the National Resources Planning Board might obtain the help of the Public Work Reserve Agency described by Melvin E. Scheidt,[20] M. Am. Soc. C. E., in October, 1941, the reply was that the Public Work Reserve Agency was "merely advisory to State and local governments on matters of procedure."

The appointment of a federal coordinator of transportation in the person of Mr. Eastman, Chairman of the Interstate Commerce Commission, lends encouragement to the idea that broad consideration may yet be had of national transportation problems with acceptable coordination of present and future facilities. It is to be hoped that the solutions will not be based entirely on consideration of profits to any of them.

W. B. Irwin,[21] M. Am. Soc. C. E.[21a]—A broad and comprehensive presentation of current transportation conditions is contained in this paper.

It has been characteristic of the development of the United States that the more convenient and expeditious mode of transportation supplanted the less efficient. Nothing, however, has been evolved to displace the railways in economy or efficiency for mass transportation of passengers or freight.

The trend of railway transportation is to maintain the highest standard of service warranted by traffic requirements and to abandon obsolete lines as rapidly as the requisite authority can be secured. The former has achieved

[20] "The Public Work Reserve," by Melvin E. Scheidt, *Civil Engineering*, October, 1941, Fig. 1, p. 577.

[21] Asst. to Vice-Pres., G. N. Ry., St. Paul, Minn.

[21a] Received by the Secretary March 18, 1942.

marked success, as evidenced by the prompt movement of large numbers of troops and their equipment, and the carrying of the heaviest freight traffic in history during 1941. The abandonment of obsolete lines has not met with equal success chiefly because of opposition to discontinuing the operation of unprofitable branches.

In discussing "Railway Transportation" the author states "Track mileage reached a maximum of 429,883 miles in 1930, declining to 408,350 miles in 1939; thousands of miles of branch and feeder lines have become obsolete." The reduction of 21,533 miles of railroad from the peak mileage is by no means a complete measure of the mileage that has become obsolete; abandonments had begun before the peak mileage was reached in 1930 and are still continuing, with many miles of obsolete line yet in operation which, for the purpose of this discussion, are considered surplus mileage in the sense of unprofitable operation.

It is of interest to consider the reasons for surplus railroad mileage, some of which are as follows:

(*a*) Competitive building of mileage in an area requiring the service of only one railroad;

(*b*) An overbuilding of railroad mileage in areas where the productive capacity of the region failed to develop sufficient traffic to support the operation of a railroad, or where depletion of the natural resources left sparsely settled districts in the wake of the exhaustion of forest areas and mineral deposits; and

(*c*) Reduced utilization of railway service due to the introduction of other transportation agencies.

The provision of the Transportation Act of 1920, requiring the issuance of a certificate of public convenience and necessity for railway construction, virtually put an end to competitive railroad building in excess of transportation needs, but considerable of such mileage remains in unprofitable operation.

Many miles of branch lines were built in agricultural areas, providing a network of railroads within the practicable limit of team-haul. This resulted in a greater railroad mileage than could be operated profitably from the production of the region. In some areas prolonged periods of drought have so reduced the production of grain and livestock that the railroads serving them must operate at a loss. This is also true of branch lines that are required to continue serving communities engaging in limited agricultural production in regions formerly devoted to lumbering or mining operations which have been exhausted.

In productive regions, where highways permit all-weather motor-truck operation, most of the livestock is trucked to market along with a considerable quantity of other farm products; less-than-carload freight, petroleum products, and some coal are trucked from distributing centers to towns on the branch lines, leaving in general only carload shipments of livestock and grain to be handled by rail to terminal markets, and a limited quantity of inbound carload freight to be transported by rail, resulting in unprofitable operation. Improved highways and the use of the motorbus and private automobile have caused local passenger traffic on the railways to diminish almost to the vanishing point.

Although a great many branch or feeder lines are unable to produce any net revenue when only their own operations are concerned, it is generally held that, if some profit is shown when the contribution of the traffic of these lines to the railway system as a whole is considered, the continued operation of the line in question is in the public interest.

In some cases where light traffic or other secondary lines of competing railways were parallel or served substantially the same areas, it has been possible to reduce the surplus mileage by securing a certificate of public convenience and necessity authorizing the abandonment and removal of one of the lines and permitting its owner to operate its trains on the remaining line under a grant of trackage rights.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

ANALYTICAL AND EXPERIMENTAL METHODS IN ENGINEERING SEISMOLOGY

Discussion

By Messrs. George R. Rich, N. J. Hoff, and Merit P. White

George R. Rich,[18] M. Am. Soc. C. E.[18a]—In modern seismic design, the use of arbitrary inertia loadings acting at the center of gravity of the structure has been superseded properly by the application of ground motions fairly representative of earthquakes to be expected at the site. Additional important economies and improved distribution of material still remain to be accomplished by increased recognition of the mitigating factors outlined by the author: First, the earthquake shock is transitory and the particular pattern required for resonance is generally repeated for only a few cycles; second, the effect of damping is controlling in cases approaching the resonant condition; and, third, a part of the seismic shock is dissipated in the propagation of elastic waves in the foundation due to the motion of the structure. Improved analytical technique is a distinct aid to progress along these lines.

Operational methods somewhat similar to the one used by the author in obtaining the seismic spectrum were evolved originally by Oliver Heaviside for studying the effect of transient electrical impulses and may be extended profitably, by means of the theory of functions of a complex variable,[19,20] in analyzing the effect of earthquake transients upon structures. These methods not only effect a great economy of time and labor by their directness and automatic elimination of integration constants, but they also afford a ready means of depicting the motion of the structure after the forced ground disturbance has been suppressed. In rigid structures with damping, this feature may be important since the response lags the transient driving impulse. Maximum stresses in such cases might occur after the ground motion ceases.

As an illustration, suppose it is desired to impress only n cycles of a sinusoidal ground acceleration upon Fig. 1 and to include the effect of damping.

Note.—This paper by M. A. Biot, Esq., was published in January, 1942, *Proceedings*.

[18] Chf. Design Engr., TVA, Knoxville, Tenn.

[18a] Received by the Secretary March 2, 1942.

[19] "Complex Variable and Operational Calculus," by N. W. McLachlan, Cambridge Univ. Press, 1939.

[20] "Operational Methods in Mathematical Physics," by Harold Jeffreys, Cambridge Tract No. 23, Cambridge Univ. Press, 2d Ed., 1931.

The basic differential equation is:

$$m \frac{d^2u}{dt^2} + \beta \frac{du}{dt} + k\, u = m\, a \qquad (43)$$

in which β is the coefficient of damping. For convenience, let $\beta/m = 2\,A$ and $k/m = B^2$. Then

$$\frac{d^2u}{dt^2} + 2\,A \frac{du}{dt} + B^2\, u = a \qquad (44)$$

The operational form for a sustained sinusoidal ground acceleration is $\frac{\omega\, p\, a_m}{p^2 + \omega^2}$, in which a_m is the maximum value of the ground acceleration ($0.1\,g$ in Fig. 5); ω is the angular velocity of the ground motion equal to $2\,\pi/T$, in which T is the period in seconds; and p, in the best modern terminology, represents the transform parameter in the Mellin inversion theorem.[21,22] Heaviside called p the operator d/dt.

The ground acceleration is suppressed after n cycles by incorporating the shift[23] operator e^{-q} (in which, to simplify typography, $q = 2\,n\,\pi\,p/\omega$), so that the operational form for the right-hand side of the differential equation becomes $(1 - e^{-q}) \frac{\omega\, p\, a_m}{p^2 + \omega^2}$. From the basic differential equation, the operational form of u is:

$$u \supset \frac{(1 - e^{-q})\,\omega\, p\, a_m}{(p^2 + \omega^2)\,(p^2 + 2\,A\,p + B^2)} \qquad (45)$$

For the case of interest in seismic design, the zeros of $(p^2 + 2\,A\,p + B^2)$ are complex numbers, so, for convenience, let $\phi^2 = B^2 - A^2$; and then placing $i = \sqrt{-1}$:

$$u \supset \frac{(1 - e^{-q})\,\omega\, p\, a_m}{(p + i\,\omega)\,(p - i\,\omega)\,(p + A - i\,\phi)\,(p + A + i\,\phi)} \qquad (46)$$

By the Mellin inversion theorem:

$$u = \frac{\omega\, a_m}{2\,\pi\, i} \int_{c-i\infty}^{c+i\infty} \frac{(e^{zt} - e^{zv})\,dz}{(z + i\,\omega)\,(z - i\,\omega)\,(z + A - i\,\phi)\,(z + A + i\,\phi)} \qquad (47)$$

or

$$u = \frac{\omega\, a_m}{2\,\pi\, i} \left[\int_{c-i\infty}^{c+i\infty} \frac{e^{zt}\,dz}{(z + i\,\omega)\,(z - i\,\omega)\,(z + A - i\,\phi)\,(z + A + i\,\phi)} - \int_{c-i\infty}^{c+i\infty} \frac{e^{zv}\,dz}{(z + i\,\omega)\,(z - i\,\omega)\,(z + A - i\,\phi)\,(z + A + i\,\phi)} \right] \qquad (48)$$

[21] "Complex Variable and Operational Calculus," by N. W. McLachlan, Cambridge Univ. Press, 1939, p. 117.

[22] "The Theory of Fourier Integrals," by E. C. Titchmarsh, Oxford Univ. Press, 1937, pp. 7, 46.

[23] "Complex Variable and Operational Calculus," by N. W. McLachlan, Cambridge Univ. Press, 1939, p. 129.

in which, to simplify typography, $v = t - \frac{2\,n\,\pi}{\omega}$.

Since the only singularities of the integrand are simple poles $z = -\,i\,\omega$; $+\,i\,\omega$; $-\,A + i\,\phi$; $-\,A - i\,\phi$; integration along the standard Bromwich contour is equivalent to $2\,\pi\,i$ times the summation of the residues[24,25,26,27] at the poles. The value of the first integral inside the brackets, Eq. 48, accordingly is:

$$u = a_m \left\{ \frac{(B^2 - \omega^2)\sin\omega\,t - 2\,A\,\omega\cos\omega\,t}{(B^2 - \omega^2)^2 + (2\,A\,\omega)^2} + \frac{(2\,A^2 - B^2 + \omega^2)\,\omega\sin\phi\,t + 2\,A\,\phi\,\omega\cos\phi\,t}{e^{A\,t}\,\phi\,[(B^2 - \omega^2)^2 + (2\,A\,\omega)^2]} \right\} \qquad (49)$$

for $t > 0$ and $< 2\,n\,\pi/\omega$.

The value of u for all times between $t = 0$ and $t = 2\,n\,\pi/\omega$ is to be taken from this expression only, which, it will be noted, correctly gives $u = 0$ and $du/dt = 0$ when $t = 0$.

The values of u after suppression of the ground acceleration are given by the sum of both integrals within the brackets, Eq. 48:

$$u = a_m \left\{ \frac{(2\,A^2 - B^2 + \omega^2)\,\omega\sin\phi\,t + 2\,A\,\phi\,\omega\cos\phi\,t}{e^{A\,t}\,\phi\,[(B^2 - \omega^2)^2 + (2\,A\,\omega)^2]} - \frac{(2\,A^2 - B^2 + \omega^2)\,\omega\sin\left[\phi\left(t - \frac{2\,n\,\pi}{\omega}\right)\right] + 2\,A\,\phi\,\omega\cos\left[\phi\left(t - \frac{2\,n\,\pi}{\omega}\right)\right]}{e^{A\,v}\,\phi\,[(B^2 - \omega^2)^2 + (2\,A\,\omega)\ ]} \right\} \qquad (50)$$

for $t > 2\,n\,\pi/\omega$.

The use of this combined value is valid only for times greater than $t = 2\,n\,\pi/\omega$. Emphasis is placed upon this important characteristic of the use of shift operators in general.[28] The time is referred to the time of quiescence as an origin, but the use of the sum of the two integrals is correct only subsequent to $t = 2\,n\,\pi/\omega$.

The operational method is not limited to impressing sinusoidal transients. Operational forms for a wide variety of impulses are available.[29] Among these may be noted as possible components of actual accelerograms the diminishing Bessel wave $p/\sqrt{p^2 + 1}$ and the Morse dash or hammer blow $(1 - e^{-p\tau})$. Operational methods also make it possible to combine these various elementary impulse forms; for example, the Bessel wave represented by the foregoing operator may be impressed immediately following a sine wave of $n + 1/4$ cycles, fairly approximating the characteristic graph observed on

[24] "Complex Variable and Operational Calculus," by N. W. McLachlan, Cambridge Univ. Press, 1939, p. 53.

[25] "Functions of a Complex Variable," by E. T. Copson, Oxford Univ. Press, 1935, p. 117.

[26] "The Taylor Series," by P. Dienes, Oxford Univ. Press, 1931, p. 233.

[27] "The Theory of Functions," by E. C. Titchmarsh, Oxford Univ. Press, 2d Ed., 1939.

[28] "Complex Variable and Operational Calculus," by N. W. McLachlan, Cambridge Univ. Press, 1939, p. 130.

[29] *Ibid.*, p. 155.

typical accelerograms. In snythesizing ground motions in this manner, care should be taken to incorporate in the operational derivation the fact that the system is quiescent at the start of the first impulse, but in motion at the start of the second component impulse.

N. J. Hoff,[30] Esq.[30a]—Predicting the stresses in buildings due to earthquakes is probably the most complicated problem a civil engineer may encounter. In problems of stresses caused by static or steady dynamic loads the use of refined mathematical methods may be required to obtain a rigorous solution, but, as a rule, an approximate answer satisfactory for practical design purposes can be obtained by the use of only elementary mathematics. Such is not the case with earthquake vibrations. Without an involved mathematical analysis not even the order of magnitude of earthquake stresses can be predicted.

The first part of this paper presents Professor Biot's earlier complicated theoretical contributions to the solution of problems of earthquake stresses in a form easily understandable to structural engineers. The explanation of the concept of the "earthquake spectrum" is of special interest since it may lead eventually to a simple routine engineering approach to this involved problem.

Advantages of the mechanical analyzer in Fig. 2 are obvious for the determination of the "spectrum." It may be of interest to compare this device to the "shaking table" used in the experiments conducted in the Earthquake and Vibration Laboratory of Stanford University, Stanford University, Calif., where the writer had the opportunity to work under the direction of Professor Jacobsen. This shaking table is mounted on balls and is actuated by a cam and follower arrangement in such a way that its motion is a replica of the recorded horizontal motion of the ground during an earthquake. On the shaking table is mounted a single-degree-of-freedom vibrating system, the forced vibration of which is recorded. The spectrums of several earthquakes were established with the aid of the shaking table.

The general appearance of the spectrums obtained at Stanford University is very much the same as that of the curves in Professor Biot's paper. The maximum oscillator acceleration, however, was found to be about 0.5 g in the Stanford tests whereas Fig. 3 of the paper shows a peak value of about 1.1 g. It is suggested that the reason for this discrepancy is the intrinsically higher frictional damping of the shaking table. Systematic investigations at Stanford University showed that small changes in the friction have little effect upon the motion of the oscillator if the friction is comparatively high, but the effect is very marked when the friction is low. Of course, even the accelerations found at Stanford are much higher than those experienced by actual buildings during the same earthquake. This must be so, since the great majority of buildings designed to withstand a horizontal acceleration of 0.1 g only was not damaged. The reason may be found in the internal friction of the buildings and in the effects of the foundation as explained by Professor Biot. Nevertheless, a

[30] Asst. Prof., Aeronautical Eng., Polytechnic Inst. of Brooklyn, Brooklyn, N. Y.

[30a] Received by the Secretary March 2, 1942.

mechanical analyzer with low friction has theoretical advantages and a damping can always be applied to it if required.

A great advantage of Professor Biot's mechanical analyzer is its ease of operation. This is due mainly to the fact that it makes direct use of the accelerogram of the earthquake. For the investigations on the shaking table the accelerogram is integrated twice and the displacement curve so obtained is used for constructing the cam. The numerical integration is very laborious as is evident to all who ever saw an earthquake accelerogram. For this reason, Professor Martel of the California Institute of Technology suggested the use of the accelerogram for the construction of the cam. Such a procedure, however, was not found practicable since it would require a cam of prohibitively large diameter.

In the investigation of the vibrations of buildings with distributed mass and elasticity, Professor Biot made the tacit assumption that the maximum shear in the building is the greatest of the maximum shears of the different natural frequencies. In the Stanford tests it was found that in many cases the time of the occurrence of the maximum shear was approximately the same for single-degree-of-freedom models of different natural frequencies when subjected to the same ground motion. If this holds true for buildings of many degrees of freedom, the cumulative effect may cause a maximum shear greater than indicated by the spectrum. Furthermore, it is even conceivable that in such a case the maximum shear would occur somewhere else than at the base of the building. It is thought that this problem may merit some further consideration.

The calculations relative to the "whip effect" and the influence of the foundation are of great importance to practical design. It is desirable that the investigation of these problems be continued with a view to establishing a reliable and easily applicable procedure of calculation such as was achieved in the simpler cases with the aid of the concept of the earthquake spectrum. It is hoped that, in due season, Professor Biot again will be able to devote some of his time to problems of earthquake stresses in buildings.

Merit P. White,[31] Assoc. M. Am. Soc. C. E.[31a]—Professor Biot's paper is probably one of the most significant of those which have appeared in the field of engineering seismology. Although many of the ideas presented are not new, nevertheless the fact that this is the first published attempt to present a complete picture of the earthquake problem makes it important.

As in nearly every other branch of engineering, there are two possible ways to attack the problem of earthquake resistant design. One of these represents the empirical, trial-and-error school of thought which, in general, is responsible for the present design methods. Actually, this approach can give, and has given, excellent results, partly on account of what must be called "engineering intuition." Nevertheless, the writer prefers the other approach, which may be characterized as the rational approach, which attempts to isolate and to understand the significance of the different factors involved in a problem. The rational method may be based on experiment, or on a combination of experi-

[31] Asst. Prof., Illinois Inst of Technology, Chicago, Ill.

[31a] Received by the Secretary March 23, 1942.

ment and theory. Certainly, Professor Biot's paper is representative of this approach.

Mechanical Analysis of Accelerograms.—The use of the mechanical analyzer described by the author certainly will result in a saving of time when compared with the arithmetical solution for the response of an oscillator to an earthquake. However, the writer believes that the difference is not always as great as the author implies. In 1938 a comparatively rapid tabular method for solving the equation of motion was developed at the California Institute of Technology.[6] This method has the advantage that the effects of various amounts of damping can be found with little additional labor. Of course, a further advantage is the fact that such a method requires no mechanical equipment. So far as the writer knows, the first use of the mechanical analyzer for finding oscillator response to an earthquake motion was by Frank Neumann of the U. S. Coast and Geodetic Survey in 1936. In this work the earthquake displacement curve, obtained by double integration of an accelerogram, was used to govern the motion of a torsional pendulum. In 1939, Ralph E. Byrne, Jr., Jun. Am. Soc. C. E., and the writer suggested a method by which an accelerogram might be used directly to actuate a mechanical analyzer.[32] This is the principle of the author's analyzer.

Peaks Appearing in Spectrum Curve.—The physical explanation for the peaks that always appear in an earthquake spectrum is a matter of importance. If these peaks truly represent characteristics of the basic earthquake motion, then, as suggested by the author, they may account for some of the paradoxical occurrences in earthquakes. However, to the writer, a more satisfying explanation is one suggested by Hugo Benioff of the California Institute of Technology; namely, that the peaks appearing in an earthquake spectrum are due to the influence of the motion of the building housing the recorder on the record made by the recorder. To test this hypothesis, extensive forced vibration tests, in which a shaking machine was used, were made by the U. S. Coast and Geodetic Survey with the assistance of Mr. Byrne and the writer in the summer of 1938.[6] It was found: (*a*) That there was definite correlation between the natural building periods and the periods of spectrum peaks for the earthquake records made by recorders in the buildings in question; and (*b*) that forced oscillation of a building gave measurable motion not only in the basement of the structure, where recording instruments may be located, but over a surrounding area, 1,000 ft or more in extent. Hence, spectrum peaks may even be caused by neighboring structures. The building coupling effect, in which a faint but regular motion is superimposed on the basic earth motion, might not be in evidence on the seismogram, but on account of its regularity it would have a large effect on the spectrum.

Change of Building Period with Amplitude.—The author states that a change of building period with amplitude of motion is effective against resonance effect. This is generally true, although it is conceivable that the opposite

[6] "Some Studies on Earthquakes and Their Effects on Structures," by R. R. Martel and M. P. White, Rept. on Earthquake Studies for Los Angeles County, Pt. I (1939) (unpublished).

[32] "Model Studies of the Vibrations of Structures During Earthquakes," by Merit P. White and Ralph E. Byrne, Jr., *Bulletin*, Seismological Soc. of America, Vol. 29, No. 2, April, 1939, pp. 327–332.

may occur—that is, an oscillator which is slightly off resonance may acquire enough amplitude to cause a period change, putting it in better resonance. However, if the spectrum peaks are not characteristic of the basic earth motion, true resonance cannot exist, and a small change of building period will have no particular importance.

Effect of Foundation Yielding and of Non-Uniformity of Mass Distribution or Stiffness of Building.—The general case, in which the structure is not uniform and rests on a yielding foundation (but has vertical planes of symmetry, or near symmetry), can be treated quite simply as follows: To determine the maximum shearing force at the base of the structure, the frequencies and shapes of the lower modes of vibration and the distribution of weight along the height of the building are needed. No other information regarding the foundation or the characteristics of the structure is required (damping is neglected). The natural frequencies of any building are found easily by the use of vibration meters or by comparison with similar structures in similar locations. The mode shapes (usually only the fundamental is really important) ordinarily can be assumed with sufficient accuracy. The distribution of weight is easily found, of course.

As was stated by the author, the total response will be the sum of the responses of the various modes, the fundamental mode predominating. Each mode will be excited in much the same way as is a simple oscillator of the same period.

Letting $Y_n(x)$ be the shape, T_n the period, and V_n the maximum shear at the base of the building, all for the nth mode of vibration, and $\rho(x)$ the mass per unit height of structure at the height x, it can be shown that

$$V_n = A(T_n)\,\frac{\left[\int_0^h Y_n(x)\,\rho(x)\,dx\right]^2}{\int_0^h Y^2{}_n(x)\,\rho(x)\,dx} = m\,A(T_n)\,R_n \qquad (51)$$

in which

$$R_n = \frac{1}{m}\,\frac{\left[\int_0^h Y_n(x)\,\rho(x)\,dx\right]^2}{\int_0^h Y^2{}_n(x)\,\rho(x)\,dx} \qquad (52)$$

In Eq. 51, $A(T_n)$ is the ordinate of the author's acceleration spectrum at $T = T_n$, h is the height of structure, and m is the total mass of the building. Thus, R_n is the multiplying factor that gives the ratio between the maximum shear at the base, caused by the nth mode, and the maximum shear in a simple oscillator of equal mass and the same period. (Note that only the shape of $Y_n(x)$ is significant, and that the scale assumed has no effect.) Eq. 51 can be shown to reduce to Eq. 18*b* of the paper for the particular case in which weight distribution and stiffness are both constant.

Now, for the fundamental mode, consider the importance of foundation yielding and of mode shape on the numerical value of the multiplying ratio R, defined by Eq. 52, as demonstrated for the following cases:

(1) Weight concentrated at a point (the simple oscillator). Ratio $R = 1.0$.

(2) Structure has uniform stiffness and weight distribution; base is rigid. Ratio $R_1 = 0.81$.

(3) Weight is uniformly distributed, and the fundamental mode is linear as in Fig. 9 (this might represent a very stiff structure on a yielding base, a structure having a particular variation of stiffness, or one in which bending and shearing deflections are about equally important). Ratio $R_1 = 0.75$.

(4) Weight uniformly distributed; fundamental mode is parabolic as in Fig. 10 (this is approximately the shape for bending deflection). Ratio $R_1 = 0.555$.

The effect of non-uniform distribution of weight will be similar to the effect of variation of mode shape. It appears that for the fundamental mode the maximum shear generally will vary between the value corresponding to the ordinate of the acceleration spectrum and about one half this amount. It can be shown that the sum of all the R_n-values for any symmetric structure must equal unity. Hence, R_1 cannot exceed unity, which is its value in the case of the simple oscillator. Also, the smaller the value of R_1, the greater must be the remaining R-values and the more important will be the shears due to the higher modes.

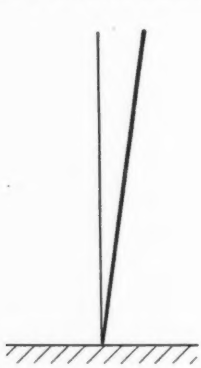


FIG. 9.—LINEAR FUNDAMENTAL MODE

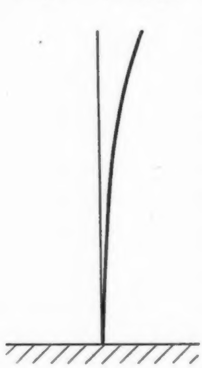


FIG. 10.—PARABOLIC FUNDAMENTAL MODE

Relative Importance of the Fundamental and the Higher Modes.—The relative importance of the fundamental and the higher modes in a particular case will depend on: (*a*) At what point of the structure shear is determined; (*b*) the values of the different R_n-values of the structure; (*c*) the periods and shapes of the different modes; and (*d*) the shape of the acceleration spectrum.

As an illustration, consider a uniform structure on a rigid base. Then, $R_1 = 0.81$; $R_2 = 0.09$; $R_3 = 0.032$ (from Table 1); and $T_1 : T_2 : T_3 = 1 : \frac{1}{3} : \frac{1}{5}$. Assume that the spectrum shape is given by Eq. 13—that is, $A(T) = \frac{0.2\,g}{T}$. Then, $A_1 : A_2 : A_3 = 1 : 3 : 5$; and, at the base of a uniform structure, $V_1 : V_2 : V_3 = 0.81 : 0.27 : 0.16$.

Variation of Shear with Elevation.—Away from the base, the situation will be somewhat different from that at the base. For example, using the same spectrum as before, at a point one third down from the roof of a uniform building, the maximum shears due to the different modes are in the ratios $V'_1 : V'_2 : V'_3 = 0.40 : 0.27 : 0.08$. Here, the second mode shear is about two thirds as great as the fundamental shear.

The effective acceleration for a section of a building, or the ratio of the maximum shear at that point to the total mass above it, is not a constant, as is

generally implied in building codes, but increases with elevation. Considering only the fundamental mode of a uniform building, the following effectiveness factors are found:

Relative distance $\frac{x}{h}$ below the roof	Effectiveness factor C'_1
0.0	1.275
0.2	1.255
0.5	1.145
0.8	0.965
1.0	0.810

Thus, the maximum acceleration at the roof of a uniform building, caused by its fundamental mode, is 1.275 times the maximum acceleration of a simple oscillator of the same period, and is $\frac{1.275}{0.81} = 1.57$ times the effective acceleration for the entire structure. This is the "whip effect" mentioned by the author. It is present in any flexible structure.

In conclusion, the writer suggests that much of the information and many of the conclusions which apply to earthquakes are also applicable to what are now even more important problems in dynamics—namely, the effects on structures of explosions, and the impact of projectiles.

Correction for *Transactions:* In January, 1942, *Proceedings*, Figs. 3, 4, 5(*b*), and 6, on pages 54 *et seq.*, change "Time, *t*, in seconds" to "Period, *T*, in seconds"; and on page 67, line 3, change "a certain function of time," to "a certain function of period."

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
DISCUSSIONS

DRAINAGE OF LEVEED AREAS IN MOUNTAINOUS VALLEYS

Discussion

By L. K. Sherman, M. Am. Soc. C. E.

L. K. Sherman,[11] M. Am. Soc. C. E.[11a]—An interesting and logical analysis of the problem of local or interior drainage of leveed areas in mountainous valleys is presented in this paper. The author calls attention to the difference between conditions in the lower Mississippi River and the Susquehanna River valleys. Be that as it may, the conditions on the upper Mississippi and the Illinois rivers differ from those on the Susquehanna River in degree rather than in principles involved. Many of the levee districts are adjacent to steep bluffs extending to a height of from 200 to 600 ft. The runoff from the lateral streams is swift and flashy.

In either location, the first principle of design is the elimination, in so far as possible, of all of the hill water entering the leveed area. In the case of bottom lands to be used as agricultural drainage and levee districts, the upstream and downstream limits are generally confined to the distance between two lateral streams. Major lateral streams, through a town or between two adjacent drainage districts, must be carried direct to the river in a wide floodway and with levees of ample height at their junction with the bluffs or hills. Many levee districts were drowned out during the floods of the Illinois and Mississippi rivers in 1926–1927 due to overtopping of the lateral levees when there was ample freeboard along the main river. The Illinois and Mississippi rivers probably have a longer, higher stage period than do the rivers in Pennsylvania. The opportunity for the occurrence of peak lateral flow with fairly high river stages is not so great in Pennsylvania. Nevertheless, the additional insurance cost is so relatively small that the design of lateral levees should be predicated upon the synchronization of peak flow in both lateral and river channels. The writer would take exception to the author's remark under "Local Drainage Requirements": "A corresponding degree of protection is not required against flows from tributaries that pass through leveed areas."

Note.—This paper by Gordon R. Williams, Assoc. M. Am. Soc. C. E., was published in January, 1942, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: January, 1942, by Merrill Bernard, M. Am. Soc. C. E.

[11] Cons. Engr., Chicago, Ill.

[11a] Received by the Secretary March 10, 1942

The author states (see heading "Methods of Disposing of Local Drainage: (1) Levees or Walls Along the Tributary Stream"): "Some tributaries along the Susquehanna * * * then flow parallel to the river, sometimes for several miles, before finally emptying into the river." This is also a common phenomenon on the Mississippi River. The flood profile of the lateral, at the upper end of the chute, is generally at a higher elevation than the main river across the dividing strip of land. In the case of the Fabius River, in Missouri, advantage of this situation was taken to dig a pilot channel cutoff across the strip between the Fabius and the Mississippi rivers. In less than two years, the floods of the Fabius had enlarged the cutoff to full size and had completely filled up the old stream channel, below the diversion, to a depth of 3 or 4 ft above ordinary low water. A serious menace and expense in levee maintenance were overcome by this simple expedient.

Without resort to excessive factors of safety, the author has presented a logical analysis of the rainfall-runoff problem for small tributary streams. Neglect to realize the importance of this problem, undoubtedly, is the reason for the numerous levee failures heretofore mentioned.

Fig. 4 may be a little puzzling to those who have become accustomed to using only the percentage distribution graph. The two hydrographs shown are true unit hydrographs. Each one represents the volume of 1 in. of runoff in specified times.

There appears to be little agreement on the definition of time of concentration. The writer defines it as the time period from the beginning of a uniform rate of rainfall on a basin until the runoff rate becomes a constant.

On this basis the writer tested the 33-min hydrograph in Fig. 4. Six average ordinates were scaled from the 10-min unit hydrograph. The summation process was applied to a series of 10-min net rains of 0.303 in. or a rate of 1.818 in. per hr. The average rate of runoff between 30 and 40 min was 940 cu ft per sec; checking a peak somewhat less than 1,000 cu ft per sec, the average between 40 and 50 min was 993 cu ft per sec and at 50 min the discharge was 1,000 cu ft per sec and continued thereafter at that rate as long as the uniform rain persisted.

This curve here developed by the writer is the "cumulative curve" introduced by Russell Morgan and D. W. Hullinghorst.[12]

A unit hydrograph, if available, reflects storage effect, and independent storage determination is not necessary; $T\,c$ is not an exact constant for a given basin. It varies with the condition of initial soil moisture and time of transit. The latter is governed by stage or rate of discharge. For all practical purposes, the author's derivation of $T\,c = 33$ min checks with the writer's definition of time of concentration.

The use of zero infiltration capacity is sound for most levee districts when the river stands against the levee. Many Illinois and upper Mississippi districts, with sandy substrata, show an actual inflow seepage of 0.25 in. or more per day. These same districts in July and August will have a large infiltration capacity.

[12] See "The Flood Hydrograph," by Howard M. Turner and Allen J. Burdoin, *Journal*, Boston Soc. of Civ. Engrs., July, 1941, discussion p. 272.

It is to be expected that the requisite pumping capacity for interior drainage should be greater for leveed municipalities than for leveed agricultural districts. Fig. 5 indicates an astounding difference. The experience of years has shown that a pumping capacity of 0.27 in. in 24 hr is adequate for all districts except those affected by continuous backwater from the river. In this case 0.50 in. in 24 hr is required.[13] A part of this difference is probably due to the availability of the interior ditches and low lying waste lands for storage, and a part is due to the fact that a material infiltration capacity exists during the season of possible crop damage.

[13] "Cost of Pumping for Drainage," by J. G. Sutton, *Technical Bulletin No. 327*, U. S. Dept. of Agriculture, October, 1932.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

PROTECTIVE AND REMEDIAL MEASURES FOR SANITARY AND PUBLIC HEALTH ENGINEERING SERVICES

Discussion

BY MESSRS. J. E. BURCHARD AND F. J. WILSON, CHARLES HAYDOCK, AND JOHN E. KIKER, JR.

J. E. BURCHARD,[5] AND F. J. WILSON,[6] MEMBERS, AM. SOC. C. E.[6a]—In this generally admirable Report, Fig. 5 is the view of a proposed trench-type shelter for personnel. Experience in Great Britain has demonstrated that this type of shelter is not satisfactory. Although it affords perhaps adequate splinter and blast resistance, it is highly sensitive to earth shock. Not only may the occupants be crushed by the actual earth movements, which may be very large, but there is also the possibility of disaster from even an explosion at moderate distance in that the earth movement may disengage the roof from the walls, thus permitting collapse. These statements are based on actual observations abroad and tests in the United States.

A bulletin, "Air Raid Shelters in Buildings,"[7] prepared by the War Department in collaboration with other federal agencies, while not recommending trench shelters, describes methods of construction where space limitations prohibit the use of other type shelters. A general preference for surface shelters has been established in Great Britain and is favored in the United States. Descriptions and typical designs are included in a revision of the bulletin, "Protective Construction, Structures Series, Bulletin No. 1."[7]

CHARLES HAYDOCK,[8] M. AM. SOC. C. E.[8a]—There are at least two quite different approaches to the important problem treated by the Committee—

NOTE.—This Progress Report of the Sanitary and Public Health Engineering Division of the National Committee of the Society, on Civilian Protection in War Time, was published in January, 1942, *Proceedings*. Discussion on this Report has appeared in *Proceedings*, as follows: March, 1942, by Donald M. Baker, M. Am. Soc. C. E.

[5] Director, Albert Farwell Bemis Foundation, Mass. Inst. Tech., Cambridge, Mass.

[6] Lt. Col., Corps of Engrs., U. S. Army, Care, Office, Chief of Engrs., Washington, D. C.

[6a] Received by the Secretary February 25, 1942.

[7] Publication pending by the Office of Civilian Defense, Washington, D. C.

[8] Cons. Engr., Philadelphia, Pa.

[8a] Received by the Secretary March 16, 1942.

namely, the technical approach adopted, which is useful to engineers and administrative officers, and a practical approach suitable to the requirements of men in the field and operators. This latter type is evidenced by the following "Safeguards for Water Supply" prepared by the writer for local distribution in December, 1941:

General.—

1. Formulate a definite emergency plan, suited to individual conditions.
2. Prepare an alternate plan, suitable to conditions if telephone and power services are interrupted.
3. In the event of an emergency, do not become excited but keep cool, act promptly, and remember that to be forewarned is to be forearmed.
4. Be certain that every member of the organization knows his duties and station in emergencies.
5. Provide a place for everything and have everything in its place when not in use.
6. If the supply is damaged, promptly consider restrictive measures to maintain a reserve for domestic uses and the fires that may follow.
7. If the water supply system is damaged, sterilize and flush the repaired section thoroughly before resuming service, particularly if sewage may have entered the mains.
8. Improve the distribution system to function in emergencies.
9. Investigate all stand-by power sources that are likely to be available in emergencies.
10. Sample the water supply frequently, and investigate any change in quality.

Personnel.—

11. Provide identification badges for all employees.
12. Know the connections of all key employees.
13. Instruct employees to report any unusual conditions and investigate them promptly. They may indicate important developments.
14. Have conveniently available telephone numbers and addresses of employees and protective and defense agencies.

Protective Agencies.—

15. Maintain contact with protective agencies, such as local and state police, and with defense agencies. Confer with the Federal Bureau of Investigation on special problems.

Communications.—

16. Consider advisability of equipping maintenance cars and isolated, critical points with police radio.
17. Provide telephones and radios for isolated, critical points.

Neighboring Systems.—

18. Contact the personnel of near-by water systems and be prepared to cooperate with them.
19. Interconnect with all adjacent water systems that meet the standards of the State Health Department. A smaller system may be able to assist a much larger system in an emergency.

Reserves.—

20. Provide reserve supplies of all essential materials, adequate for emergencies.
21. Store all emergency reserve materials where they will not be damaged by an incident that damages a critical point.
22. Save the reserves for a genuine emergency. Do not use them unless needed to maintain adequate service or unless their use will aid in the National Defense.

Critical Points.—

23. Do not publicize the critical points of the water supply system.
24. Prohibit unauthorized access to, and parking adjacent to, all critical points.
25. Erect protective fences around, and floodlight, critical structures.
26. Patrol critical points and endeavor to have police do likewise.
27. The use of airplanes is restricted. Special restrictions may be necessary for the critical points of the system.

Records.—

28. Have adequate records. Do not depend on memory.
29. Prepare multiple sets of records and keep the master sets in the safest possible place.

Recommended Reading.—

New England Water Works Association—First Report of Committee on Water Works Emergencies; 613 Statler Building, Boston, Mass.

U. S. Engineer Department, Second Corps Area—Civil Defense; Headquarters, Governors Island, N. Y.

American Municipal Association—British Cities at War; Report published by Public Administration Service, 1313 East 60th Street, Chicago, Ill.

International Association of Fire Chiefs—(*a*) A Major Disaster Emergency Plan; and (*b*) Disaster and War Emergency Planning; 24 West 40th Street, New York, N. Y.

National Fire Protection Association—Fire Defense; 60 Batterymarch Street, Boston, Mass.

National Technological Civil Protection Committee—Report of American Observers in England.

Summary.—The necessity for concise data is shown by certain air raid instructions that have been issued, to the effect that bathtubs should be filled upon the sounding of the alarm, and that unauthorized persons should operate valves in municipal water systems. It is not necessary to dwell on the unfortunate results that might follow such practises. It is not likely that such instructions would have been issued if they had been submitted to competent engineering review such as is available through the National Committee of the Society.

Difficulties in securing materials indicate that in many cases it probably will be necessary to meet the emergency conditions without emergency construction and without expanding the normal stock of repair parts.

It is to be hoped that the application of these principles may not be required, but plans should be made and, if the application of the plans becomes necessary, no doubt adequate materials will be made available promptly to meet the emergency, in order that people in the affected areas may not be deprived of the benefits of essential services.

JOHN E. KIKER, JR.,[9] M. AM. SOC. C. E.[9a]—This Report furnishes a valuable summary of an important subject and should lead to improvements in community sanitation as a measure in wartime protection. The Committee is to be commended for a job well done.

In the discussion of underground sources of water the Committee properly points out that, if a supply is comprised of only one well, arrangements should be made for additional wells in different locations in order to prevent the likelihood of total damage by a single explosion. They also point out that, unless there is more than one well, arrangements should be made for an emergency supply from some other source of water, and they mention later that a break of a single supply line will put the water works out of commission. It logically follows that the chances for putting a single supply line out of commission will be in proportion to the length of the line, and this brings up the question of extremely long lines from sources so far removed that they may not be adequately protected. The dire consequences of depending upon remotely located surface supplies that could not be protected properly are believed to have been demonstrated in the recent experiences at Hong Kong, China, and Singapore, Straits Settlements.

In constrast to these types of supplies, multiple-well systems have been developed at other places that are important from the standpoint of defense and military or naval operations. To cite just one example, the French Government engaged an American firm in 1930 to develop a multiple-well system for furnishing some 30,000,000 gal of water a day to the cities of Saigon and Cholon, French Indo-China. Some twenty odd wells were drilled and, with the system as completed, the destruction of one, or even five or six, wells would not create a serious water shortage.

It is realized that there is a limit to the practicability of well supplies, particularly in some large American cities; but it is believed that in many instances the chances for failure of water supplies may be appreciably reduced through the proper construction of wells having their equipment housed underground. Such treatment plants as are necessary could also be submerged, and this is frequently done. In view of these factors, it would seem that underground sources of supply should be developed where they are reasonably practical, or, at least, they should be given first consideration in important defense areas. In so far as the locations of new sites for defense purposes are concerned, it is believed that careful investigations should be made in advance to the end of locating the sites where ground water is available so that the water supply may be protected properly in case of attack.

[9] San. Engr., Poughkeepsie, N. Y.

[9a] Received by the Secretary March 19, 1942.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
DISCUSSIONS

UNUSUAL EVENTS AND THEIR RELATION TO FEDERAL WATER POLICIES

Discussion

BY MESSRS. EDWARD H. SARGENT, DANA M. WOOD, JAMES S. SWEET, AND J. L. CAMPBELL

EDWARD H. SARGENT,[10] M. AM. SOC. C. E.[10a]—The author is to be congratulated on his splendid paper describing the basic reasons underlying the changes in the federal water policies in the decade from 1930 to 1940.

The gates of the Conklingville Dam creating the Sacandaga Reservoir, constructed by the Hudson River Regulating District on the Sacandaga River, the principal tributary of the upper Hudson River, were first closed on March 27, 1930, impounding that spring's flood waters. This reservoir has an available capacity of 760,000 acre-ft, and release of the stored water for the regulation of the flow of the Hudson River for the public welfare, including navigation, sanitary improvement, and power, began on July 1, 1930.

With reference to the author's statement (see heading "Multiple-Use Projects") that, "Since 1930 there has been a profound change in federal participation in multiple-use water projects," the writer feels that the success of the Sacandaga Reservoir in achieving these multiple purposes contributed to the changing of the current engineering opinion regarding such functioning.

DANA M. WOOD,[11] M. AM. SOC. C. E.[11a]—This excellent paper rightly stresses the impetus given to multi-purpose planning by quite recent events. However, the trend has existed even from the earliest days of the Nation, and was strong long before the past decade. Also, this paper is based largely upon the effects of Congressional acts, whereas innumerable court decisions also have had a very definite part in broadening federal water policies. Of course, both the Congress and the courts consist of men influenced by the social impacts of droughts and floods, as well as economic and war stresses. All these factors have been present from earliest days.

NOTE.—This paper by W G. Hoyt, M. Am. Soc. C. E., was published in February, 1942, *Proceedings*.
[10] Chf. Engr., Hudson River Regulating Dist., Albany, N. Y.
[10a] Received by the Secretary February 17, 1942.
[11] Head Hydr. Engr., TVA, Knoxville, Tenn.
[11a] Received by the Secretary March 11, 1942.

At least two other papers should be read along with this one to clarify the broad background for present federal water policies. In 1938, James L. Fly[12] presented the legal development of federal policies with respect to navigation, flood control, irrigation, and finally planning for multi-purposes. Further progress has been made since that date. David E. Lilienthal and Robert H. Marquis, in 1941,[13] emphasized the expansion of governmental business activities and the advantages of the government corporation for the administration and development of multi-purpose projects. Both papers definitely indicate why recent rapid advances in policies and accomplishments have been possible.

One reason why so much progress could be made during the past decade may lie in the tremendous advance made in the arts during the previous two decades. Engineers were prepared to accomplish great things when the need became urgent.

The writer had occasion recently to ask the U. S. Weather Bureau for its definition of a drought, and was much surprised to learn that the best it had to offer, in a hurry, was "any time when there is insufficient moisture to support the continued growth of vegetation." The author lists several years during which critically low runoffs occurred in different areas. In the comprehensive development of any single, large watershed, it will be found that consideration must be given to critically low periods lasting two years, three years, or even longer. This is important with respect to hold-over storage problems from one year to another during such periods. Considering the high degree of regulation possible after 1945 in the Tennessee Valley, indications (in advance of complete runoff records) are that the most severe drought of record has lasted from 1939 into the current year (1942), with the 2-yr period 1930–1931 holding second place, and a drought in 1903–1905 holding third place for long periods. The year 1925 was the most critical single year for storage requirements in this area and is probably controlling, even when compared with the longer periods, for regulation resulting from multi-purpose objectives. However, if operations were for maximum power production, the longer periods would probably be controlling. This is mentioned to call attention to the fact that what may be considered as a critical drought with one objective in view may be defined quite differently for other objectives.

A similar argument might be applied to floods. Regulation which might not be fully effective in a maximum flood would certainly affect the frequency of occurrence of lesser floods as well as their magnitudes.

The important effect of broadening federal water policies with respect to both droughts and floods is to lessen their damages by adequate planning on a large scale, beyond the ability of private capital or private objectives to accomplish. Since federal planning should accomplish as many social benefits as possible, many objectives should be included and the necessary compromises made between them. In the TVA this is done through the extensive use of operating rule curves for all projects, modified from time to time as experience

[12] "The Rôle of the Federal Government in the Conservation and Utilization of Water Resources," by James Lawrence Fly, *Pennsylvania Law Review*, Vol. 86, No. 3, January, 1938.

[13] "The Conduct of Business Enterprises by the Federal Government," by David E. Lilienthal and Robert H. Marquis, *Harvard Law Review*, Vol. 54, No. 4, February, 1941.

is gained. Herein lies one of the important jobs for the engineer, mentioned by the author.

Any mention of the multi-purpose objectives of federal agencies, in the case of southern projects in particular, should include that of malaria control. The TVA, in cooperation with the U. S. Department of Public Health and local agencies, has worked on this problem from the beginning of its activities. The creation of large bodies of water where none existed previously, with associated problems of proper beach drainage and control of vegetation along the shores, places the responsibility for controlling the breeding of mosquitos directly upon the federal agency that developed the project. It is not a minor problem.

One problem that will offer increasing difficulties in its solution is that of coordinating the work of the various federal agencies engaged upon water control and development, to prevent duplication of effort; and to establish procedures for the exchange of experience data and engineering information, and perhaps even of personnel at a time when it would be most helpful. The centralized control that is so often suggested as suited to a federal Public Works Department is directly opposed to the benefits obtainable under decentralized agencies working closely with local state agencies, the people and institutions most affected, but with proper federal checks and controls. The engineering profession could be helpful in determining the more efficient administrative theory.

If the second scheme is adopted ultimately, it will give rise to a further problem—how to make the experience and improved techniques of one agency promptly available to another. Publication of technical information is not the entire answer because frequently it is too long delayed; it is not distributed or made available where it might be most helpful either because of cost or not knowing where the need exists. There is too much published now for any one man to review thoroughly along with his active work. Engineers frequently are not interesting and concise writers; and some of the busiest men, whose ideas and experiences might be of the greatest value, cannot or will not give the necessary thought and time required to the preparation of a paper or book along with their required daily work. Perhaps, too, progress and changes are sometimes so rapid that some engineers hesitate to commit themselves in print. There are a few who do not wish to give away their "stock in trade"—a very mistaken idea, professionally.

Perhaps committees in federal engineering groups could be formed to function something like those in the former National Electric Light Association. Annual summaries of progress and experience might be issued on classified subjects, preceded by a comprehensive review of background material. If this procedure proved successful in private practice, it could be made equally so among federal agencies. The National Resources Planning Board might be the means of accomplishing this end.

James S. Sweet,[14] Assoc. M. Am. Soc. C. E.[14a]—A few papers have been written in the past describing hydraulic activities of the government; but none

[14] Regional Hydrologic Engr., U. S. Weather Bureau, La Guardia Field, N. Y.

[14a] Received by the Secretary March 18, 1942.

attempted to relate influences of social events and natural phenomena to the federal water policies. From this standpoint the paper by Mr. Hoyt occupies a unique place in the literature on the subject. Not only is the description interesting, but it also clarifies sometimes unexplainable tendencies of the national legislation in the hydraulic field. Like other writers on this subject, the author fails to indicate weak spots of the national water policy, or to offer some constructive criticism. Even the restrictions of a federal position would not bar such criticism, if it is not malicious.

Throughout his paper the author mentions various government departments in which hydraulic or hydrologic activities exist. To a layman this fact gives reason to believe that all these activities are, or could be, coordinated. Unfortunately, this fact in itself is a sign of inefficiency, for, under the law, the jurisdiction over these activities is given to the respective departments where they are placed. This makes coordination difficult. Hence, there is much duplication and delay. An attempt toward a coordinated action was made by appointment of the Water Resources Committee of the National Resources Planning Board. However, this committee has no administrative powers; and its voice, although often speaking truth and wisdom, is unheard by those to whom it is directed.

Placement of all hydrologic activities of the federal government in one bureau would be impracticable under the present circumstances. As an alternative, creation of a hydraulic coordinating joint committee seems quite feasible. This committee should consist of the executives of all branches of the government dealing with hydraulic works and hydrologic data. It should have a revolving chairmanship. It should act not only in the advisory capacity, but should also have executive powers. This would be facilitated by the fact that the head of each agency involved would serve on the committee and be its chairman at one time or another.

This joint committee should fill in gaps of inadequacy, eliminate overlapping activities, and coordinate existing and planned work of all hydrologic agencies, thus promoting efficiency and economy.

J. L. Campbell,[15] M. Am. Soc. C. E.[15a]—The federal water policies developed since 1930 constitute one phase of a course of the federal government away from that American way of life which rests upon and is sustained by private initiative and enterprise toward some form of paternalism wherein, eventually, the conception that government is the servant of the people would perish, and the people would become dependents upon and servants of government. That is a momentous issue raised by some of the policies of the new order known as the New Deal.

In the first decade of such departure, beginning in 1932, certain phases of the broad issue indicated came to the fore in attempted regimentation of some of the economic life of the people by certain programs of the National Industrial Recovery Administration and the Agricultural Adjustment Administration—creatures of the new order—wherein it became evident that the people

[15] Chf. Engr., Northwestern Pac. R. R. (*Retired*), Oakland, Calif.

[15a] Received by the Secretary March 2, 1942.

involved would not unanimously and completely voluntarily accept and comply with said programs, and that, consequently, basic objectives of the new order could be brought to full fruition only by dictatorial compulsion of all the people concerned. The clear implication of that situation is that certain ways of life in the new order could be made completely effective and successful only by government by dictatorship. In substance, that issue was brought to the Supreme Court of the United States, and the Court set up a bar against dictatorship; but that does not finally dispose of the general issue. Since that decision, the personnel of the Court has undergone an unprecedented and revolutionary change.

The activities to date of the Tennessee Valley Authority point to elimination or subordination of private initiative and enterprise and substitution of government ownership and operation. Similar implications lie in the Columbia Basin Project—the Grand Coulee extravaganza—and in other projects of the new order.

In general, projects coming to life under the federal water policies have a common characteristic vividly illustrated in Grand Coulee Dam which came to the Congress as an hydroelectric project asking for only $65,000,000—a modest New Deal sum. Then, like the beanstalk in "Jack and the Beanstalk," this project grew rapidly into the "stratosphere" and is now approaching completion as a multiple-use project costing nearly $400,000,000. This project is without the economic justification upon which private initiative and enterprise must rest in expenditure of money. Such projects have helped mightily in doubling the national debt of 1930.

The Grand Coulee project includes proposed reclamation of more than a million acres of desert land lying above the high-water line of the reservoir created by the dam. This requires lifting the water for irrigation several hundred feet by hydroelectric power generated at the dam. This makes Grand Coulee reclamation costly and beyond the resources of the farmers. That land can be so reclaimed and farmed only if the federal government carries for the farmers a large part of the financial load. The resulting cost of food and clothing materials so produced will be above like costs in the Mississippi valley and the United States in general.

Grand Coulee reclamation was put forward and appropriations therefor were made during the time when the federal government was proclaiming an alleged overproduction of farm products, was putting the farmers into strait jackets, and was expending millions of dollars in retirement of farm lands from production for the purpose of reducing agricultural products. The barns and granaries of the country are still overflowing with farm surpluses.

Grand Coulee illustrates extremes to which public appropriations and expenditures run under compulsion of political and other pressure groups in the unusual events of the federal water policies. It is also indicative of what comes out of the whole of the new-order philosophy. Now, more than ever, unnecessary or deferable expenditures should cease during the war.

Those who support continuance of such expenditures allege that projects of the kind indicated are necessary to the war program because of the hydroelectric power which they would provide. Accordingly, immediate authoriza-

tion of construction of the St. Lawrence Seaway—a five hundred million dollar project—is urged, notwithstanding the fact that several years would elapse before power would flow from it.

If power for the war is lacking, the lack should be eliminated in the shortest way possible. Therein, time rather than cost is most vital. The quickest way is by construction of steam plants at points of power consumption. Generally, first cost of steam power is far less than cost of hydroelectric power. Moreover, in most of the war power centers of the country, there would be little difference in over-all costs of construction and operation as between up-to-date steam plants and hydroelectric plants. Finally, it would be difficult indeed for the enemy to find and destroy a multitude of relatively small steam plants, and comparatively easy to locate and destroy a few great hydroelectric plants.

Is there such lack of patriotism that pet schemes dressed in the guise of war need are delaying and crippling war effort?

A general effect of some new-order policies is discouragement of self-reliance, initiative and enterprise, and encouragement of dependence upon government for social security. That is not an American, or a healthful, trend.

Another regrettable development is the great expansion in number and size of governmental functions and agencies resulting from new-order policies and activities, upon which it is now necessary to superimpose the vast war work organizations of the government. The President suggests that many people in Washington, D. C., not in government service, should move out to give the government elbow room. Liquidation of a large number of new-order agencies would be a better remedy.

Of course, America must move forward; but she should so move in the right way, under the Bill of Rights of the individual in the charter of American liberty. A new order should not be condemned, or swallowed whole, merely because it is new. By its fruits, it should be judged. By that measure, the new order herein under criticism is partly good and partly bad. To be worthy to endure and able to serve sanely and beneficently, this new order, to the extent that it may be afflicted by them, must be purged of faults and dangers, such as are indicated in this critique.

Turning to the engineering and construction work of building the projects fathered by the federal water policies originated since 1930, the picture is a happy one wherein superb engineering design and direction and marvelous construction work have created sound and efficient structures of unprecedented magnitude and highest excellence. The engineering profession and the construction art may look in great complacence upon such achievements; but such structures are not, in themselves, justification for their existence.

Yes, under the federal water policies, there is a horn of plenty of "unusual events."

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

REDUCTION OF MINERAL CONTENT IN WATER WITH ORGANIC ZEOLITES

Discussion

BY ROBERT SPURR WESTON, M. AM. SOC. C. E.

ROBERT SPURR WESTON,[5] M. AM. SOC. C. E.[5a]—The excellent paper by Mr. Goudey shows how much the art of water softening has developed since Thomas Clark of Scotland, about a century ago, patented the process of softening by adding lime water, and J. H. Porter, more than fifty years ago, introduced its sequel, the lime-soda process, at Southampton, England.

Both of these processes have been modified and improved by better arrangements and devices for the coagulation of precipitates and by the use of carbon dioxide and chemical dispersers to prevent the accumulation of carbonates on filter sands and in distribution systems; but even the best of these modified processes has its limits, and reductions of hardness below 80 ppm have been impracticable and difficult in most cases.

The writer first heard of base exchange softening at a conference in Breslau, Germany, when the work of Werner Gans[6] was mentioned in connection with that city's deferrization problem. Since then, the development of zeolite softening has been extensive, and would be wider if communities would only realize, as has the Los Angeles district, that large savings in soap and washed fabrics, and less depreciation of water-using devices, will follow the use of softer water.

Since his first experience with softening by natural zeolite at McKees Rocks, Pa., in 1924,[7] and his later experience at South Orange, N. J.,[8] and elsewhere, the writer has had almost no municipal cases where the residual compounds were in high enough concentrations to be intolerable. He has realized how great the need is for some method of lessening these concentrations

NOTE.—This paper by R. F. Goudey, M. Am. Soc. C. E., was published in February, 1942, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: February, 1942, by Messrs. James M. Montgomery, and George L. Davenport, Jr.

[5] Cons. Engr. (Weston & Sampson), Boston, Mass.

[5a] Received by the Secretary February 19, 1942.

[6] "Verbesserung von Trinkwasser u. Gebrauchswasser für hausliche u. gewerbliche Zwecke durch Alluminat-Silicate oder künstliche Zeolithe," by Werner Gans, *Mitteilungen aus der Königlichen Prüfungsanstalt für Wasserversorgung u. Abwasserreinigung*, Berlin, 1907, Heft 8.

[7] *Journal*, A. W. W. A., Vol. 32 (1940), p. 85.

[8] *Ibid.*, Vol. 33 (1941), p. 255.

however, notably compounds of sodium in many waters which have initial hardnesses of 500 ppm or more.

Following zeolite treatment came the combination treatment with lime and zeolite which enabled one to reduce, to almost any desired degree, the hardness of waters high in sulfates.

Then, early in 1938, there appeared on the market anion and cation exchangers by the use of which, either separately or in combination, not only the hardness but the sodium content could be reduced to zero and the total solids greatly reduced as well. An excellent classification of these zeolites appeared in the author's former paper (7).[8a]

It seems almost fanciful that there are durable, organic cation exchanging zeolites on the market which will remove hardening carbonates and bicarbonates in the form of carbon dioxide gas and which may be regenerated with sulfuric acid, which, when combined with calcium, is a superlative water hardener; or, that there are anion exchangers which will remove not only bicarbonates but sulfates, chlorides, and fluorides, with increased acidity of effluent, and which require sodium hydroxide for regeneration.

Therefore, particularly valuable is the description of the author's research work and the clear exposition of the reactions involved. Interesting indeed is the showing that hydrogen zeolites may be used to remove either calcium or magnesium, or both, and that both the hydrogen and sodium cycles may be used on portions of the same water and the two mixed afterward.

All of the great possibilities that have been made available through the improvements in the durability, economy, and efficiency of the organic zeolites should make it possible to say farewell to turbid artificial ice and to boiler scale, to wash windows with waters of the desert, and to wash wool with waters from deep wells.

One rather shrinks from handling sulfuric acid or caustic soda as a regenerator because of the extreme aggressiveness of each in comparison with brine. However, resistant apparatus is available and may be used for successful handling where efficiency and cost are at stake.

One also asks if some of the regenerating chemicals may not be saved and reused, as is part of the brine at South Orange (3), with consequent material saving in cost.

In closing, the writer pleads for a simpler expression of the salt consumption for regenerating zeolites than is involved in the use of the term "grains"; for example, the expression "1.2 lb of hardness per cubic foot" seems simpler than "8,400 grains per cubic foot"—especially to those who think of hardness in terms of parts per million or pounds per million gallons. Likewise, for salt consumption it would be simpler to state "3.0 lb, or parts, of salt per pound, or part, of hardness" than "0.42 lb per 1,000 grains." It is realized, however, that those who treat boiler waters have thought in "grains" for more than half a century.

[8a] Numerals in parentheses, thus: (21), refer to corresponding items in the Bibliography (Appendix) of the paper.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

TECHNIQUE OF DETERMINING SHEARING STRENGTH OF SOILS

PROGRESS REPORT OF A SPECIAL COMMITTEE OF THE SOIL MECHANICS AND FOUNDATIONS DIVISION ON THE TECHNIQUE OF SOIL TESTS

Discussion

BY CLOYD D. BEERUP, ASSOC. M. AM. SOC. C. E.

CLOYD D. BEERUP,[2] ASSOC. M. AM. SOC. C. E.[2a]—The committee should be commended for its excellent summary of shear-test data. Such a consolidation of information helps to give a clearer picture of methods used in shear determinations and indicates many weaknesses in the present methods.

The writer agrees with the committee that the direct-shear device tends to distribute the load more evenly over the plane of failure. No doubt the number and spacing of teeth materially affect the load distribution within the sample.

Triaxial testing probably gives the best clue to the capacity of a soil to withstand stresses; but, as stated in the report, the use of rigid plates on the top and bottom of the specimen tends to restrict its movement to a lateral direction. The unequal distribution of stresses thus set up probably does not allow an accurate interpretation of results.

Although the committee covered the field of shear tests, it might have been well to consolidate some facts on methods of soil sampling. Without question, many test results should be discarded on the basis of poor sampling practices.

The writer cannot agree with the criticism that the loading equipment was "amateurish in appearance and construction." Although this statement is no doubt very true, it must be remembered that the development of good equipment depends largely on trial and error. Certainly no one would have expected an automobile manufacturer in 1900 to produce, directly, the streamlined car

NOTE.—This Report was published in February, 1942, *Proceedings*. Discussion on this Report has appeared in *Proceedings*, as follows: February, 1942, by Glennon Gilboy, Assoc. M. Am. Soc. C. E.

[2] Asst. Engr., TVA, Knoxville, Tenn.

[2a] Received by the Secretary February 26, 1942.

of today. Only through the use of "water cans, wires, bicycle wheels, hanging weights, levers, pulleys, and wooden framework" can modern testing apparatus be developed.

Considerable progress has been made in the development of testing apparatus for design purposes, but a positive means of measuring stresses in large masses of soil is apparently lacking. The engineer should be able to correlate design assumptions with actual field tests in order to eliminate costly errors. This is certainly a field for active research.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

TIMBER FRICTION PILE FOUNDATIONS

Discussion

By Messrs. Glenn B. Woodruff, Jacob Feld, G. G. Greulich, and G. S. Paxson

Glenn B. Woodruff,[5] M. Am. Soc. C. E.[5a]—Many engineers have been aware of the general principles demonstrated by this paper. The thanks of the profession are due the author for presenting the results of these tests in a manner that gives a mathematical statement of these principles.

Although Table 5 shows a remarkable agreement between theoretical and actual results, the author probably will agree that delta deposits are generally so heterogeneous that extreme precision in predicting results is unwarranted. It appears permissible therefore to simplify several of the equations by eliminating terms that will have minor influence on the results.

The load tests were made shortly after the piles were driven. Possibly, over a long period of years, the distribution of the loads from the piles to the surrounding soil will change materially. In most silty soils, the process of soil consolidation under its own weight is still continuing. If the strata near the surface are settling, these strata, rather than taking load from the piles, will be supported partly by the piles and will add to the pile load. It will be difficult to separate the results of this action from the settlements resulting from the consolidation of the strata below the pile tips, but it is to be hoped that the owners of the structures to be built will arrange for such a record of settlements under piers of varying size that certain conclusions can be drawn.

Tables 8 and 9 show clearly the advantages of wide pile spacing. This wide spacing has the disadvantage that larger, and consequently heavier, footings are required to distribute the superstructure loads to the piles. In some cases, it becomes advisable to use battered piles to combine the advantages of a small pile cap and a wide distribution of the load. In this way, each of the piles in a comparatively large cluster may be made to approach the load-carrying capacity of a single pile.

Note.—This paper by Frank M. Masters, M. Am. Soc. C. E., was published in November, 1941, *Proceedings*. Discussion on this paper was published in *Proceedings*, as follows: February, 1942, by E. H. Connor, M. Am. Soc. C. E.

[5] Cons. Engr., Berkeley, Calif.

[5a] Received by the Secretary February 27, 1942.

JACOB FELD,[6] M. AM. SOC. C. E.[6a]—A complete solution of the subject of friction piles has been presented in this paper, and the author is to be complimented for his derivation of empirical tables for practical use in design.

The average frictional resistance along the surface of a single pile at failure for timber piles (Table 4) is given as 778 lb per sq ft, as against 558 for parallel-sided concrete piles. The discrepancy is merely the result of an assumption. Shear failure in the case of either wood or concrete piles is not along the pile surface but along a soil surface, formed by the adhesion of some soil to the pile. In the case of tapered wood piles, the proper surface area seems to be governed by the maximum pile diameter, not by the average. Certainly at failure, the soil directly below the tapered surface is compressed sufficiently to act as part of the pile. Table 11 shows the data of Table 4(*a*) with such adjustment; the average soil friction for the wood piles is 600 lb per sq ft of cylinder surface, with the butt diameter.

TABLE 11.—SKIN FRICTIONS, INDIVIDUAL, TAPERED, TIMBER PILES (SEE TABLE 4(*a*))

Pile No.	Length (ft)	Load at failure (tons)	DIAMETERS (IN.)		SKIN FRICTION, IN LB PER SQ FT	
			Tip	Butt	Average diameter	Maximum diameter
T1	52	66.5	8.5	17.5	685	560
T10	59.5	96.5	11.5	17.75	900	700
T20	85	100	8.0	17.5	680	500
T34	83	110	8.5	16.75	777	600
T18	50.5	70	8.5	16.5	847	640
Average					778	600

Since the minimum value occurs with the minimum tip and the maximum value occurs with the maximum tip, there is certainly some variation with tip size. The number of test values is not sufficient to draw more definite conclusions.

A group of friction piles in itself is a means of transferring loads to lower soil layers, very much in the manner of a spread footing. The similarity of the two rules in Example 2 with the conclusions drawn from tests with spread footings, therefore, is not surprising. Only by exposure of loaded surface to soil not carrying its full capacity can loads be carried safely.

The writer has used a very rough rule for the ratio of load-carrying capacity of a group of piles to that of a single pile. Since it seems to have some relation to the result presented in this paper, the rule is given herein. The value of a pile in a group is the value of a single pile less one sixteenth for each adjacent diagonal or row pile (see Table 12). In view of the unknown relative effects of tip area and length upon the unit bearing value of a pile, the suggested reduction in pile values of a group is an easy approximation. A pile fully enclosed in a group, as the center pile in a square array of nine piles, has a value of 50% of a single pile. A minimum pile spacing distance is assumed.

This method, applied to the author's examples in Table 6 and Fig. 8, gives values of 10.25 and 10.75 equivalent single pile units for the 15-pile and 16-pile groups. This would indicate that in the 15-pile group an individual pile would

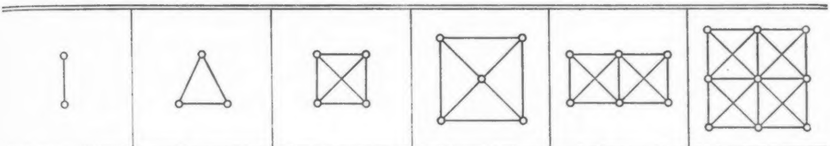
[6] Cons. Engr., New York, N. Y.

[6a] Received by the Secretary March 5, 1942.

fail at about 60 tons, if the average failure load equals 41.4 tons. This value is consistent with the individual failure tests. Fair agreement is also found with the data in Table 5.

TABLE 12.—EQUIVALENT PILE VALUES OF CLUSTERS

(Total Value of Cluster Given in Parentheses, in Terms of a Single Pile Value)



2 Piles @ $\frac{15}{16}$	3 Piles @ $\frac{15}{16}$	4 Piles @ $\frac{13}{16}$	5 Piles 4 @ $\frac{13}{16}$ 1 @ $\frac{12}{16}$	6 Piles 4 @ $\frac{13}{16}$ 2 @ $\frac{11}{16}$	9 Piles[a] 4 @ $\frac{13}{16}$ 4 @ $\frac{11}{16}$ 1 @ $\frac{8}{16}$
(1.87)	(2.81)	(3.25)	(4.00)	(4.62)	(6.50)

NOTE:—Similarly for Other Clusters.

[a] Omitting Center Pile of the 9-Pile Cluster Does Not Affect Bearing Value of the Cluster.

G. G. GREULICH,[7] ASSOC. M. AM. SOC. C. E.[7a]—Careful and adequate driving and load tests made preliminary to the final preparation of a design are described in this paper. It is apparent that, after reviewing all available recorded data, previously expounded theories and formulas were expanded and modified in an intelligent manner so as to be theoretically compatible with the test results. The closeness of the approximations is shown by Table 5. However, the close agreement shown in the table may be due, in a large measure, to the exceptionally uniform soil conditions prevailing in this portion of the Lower Mississippi Valley. In areas where more heterogeneous soils may be encountered, it is doubted whether such close agreement would be found.

The theoretical failure loads given in Table 5 are remarkably close to the actual failure loads and demonstrate that, on occasion, the law of averages will take care of widely varying factors. Reference to Table 1 shows that there is very little relationship between the hammer blows for the last foot of penetration and the actual skin friction values developed at failure. For example, in the T. & P. tests, pile T18, requiring 25 blows for the last foot of 50.5 ft of penetration, shows a skin friction value of 847 lb per sq ft, as compared with 610 lb per sq ft for T14–T17, where the number of blows for the last foot of approximately the same length of penetration (50 ft) was 27, or two more than for the much higher load developed by the first pile. Reference to the N. O. T. & M. tests, pile T34, requiring 8 blows for the last foot of 83 ft of penetration, shows a skin friction value of 777 lb per sq ft, as compared with 680 lb per sq ft for T20, where the number of blows for the last foot of 85 ft of penetration was 30.

[7] Sales Representative, Specialty Sales Div., Carnegie-Illinois Steel Corp., Pittsburgh, Pa.

[7a] Received by the Secretary March 6, 1942.

In spite of the wide variation of skin friction values for individual piles, the average value from Table 4 apparently gave exceptionally accurate results when used by Mr. Masters in predicting theoretical group behavior. This is especially notable in view of the normal and often wide variation found in the shape of timber piles and their behavior in driving when driven in isolated groups. The writer wonders if this method will prove as accurate when applied to groups at other locations. Only additional similar test data will develop this point. In the meantime, it has been demonstrated again that, having access to carefully recorded test data, it is possible to develop rules or formulas that do give reasonable results for any areas of ground which are comprehensively covered by test data.

In the writer's opinion, Mr. Masters has done a great service to the profession by the incorporation of Table 8. Perusal of this table should give any practicing engineer a vivid picture of the rapid reduction in load capacities of piles when used closely spaced in large groups. This table throws light on a remark made some years back by an elderly engineer whose name the writer cannot recall. In his practice, he had built a number of swing bridges with the main center pier founded on piles. In spite of the fact that the timber piles he ordinarily used tested satisfactorily in clusters of four for safe loads of 16 to 18 tons per pile, he found that, in order to keep settlements within workable limits for his movable structures, he could not load the piles to more than 7.5 to 8 tons.

Referring to Table 8, it will be noted that, for 2.5-ft spacing of 40-ft piles, the failure load for clusters of four is computed as 32 tons per pile. On such a pile, a 16-ton to 18-ton load might ordinarily be considered a safe working load. Going on down the tabulation to 16.5-ton load per pile, which would represent a safe load of 7.5 to 8 tons, it is noted that a group of 49 piles would have this theoretical capacity per pile:

> Multiplying 49 piles by 6.25 sq ft of contributory area, on 2.5-ft spacing, the resulting gross area is approximately 306 sq ft.

This area could well have been the area of the main piers of some of the early, light, swing bridges of this type which were approximately 20 ft in diameter. As stated before, it is believed that this paper gives a vivid explanation of the elderly engineer's remarks, and emphasizes the necessity for drastic reduction in assumed design loads for individual friction piles when closely spaced in large clusters.

G. S. Paxson,[8] Assoc. M. Am. Soc. C. E.[8a]—There is probably no part of structural design less understood than foundations, especially foundations supported by friction piles. For many years, it has been known that the value of a group of piles was not the sum of the values of the individual piles taken separately. The method of design proposed by Mr. Masters provides a quantitative solution of the problem. The remarkable correspondence between the theoretical and actual loads at failure in the Morganza Floodway tests gives great support to the proposed method as a practical design tool.

[8] Bridge Engr., State Highway Comm., Salem, Ore.

[8a] Received by the Secretary March 16, 1942.

The writer was particularly interested in the approach that the paper provides to the question of piles versus spread footings on cohesive soils. There are many contradictory statements in current engineering literature on this subject, but, so far as the writer knows, no quantitative comparison has been given.

In Fig. 12, showing the approximate vertical pressure, q_v, contours are given for a foundation supported on piles and for the same foundation resting directly on the soil. The pressure contours on the left-hand side of the figure show the pressures in pounds per square foot for the foundation resting directly on the soil, whereas those on the right-hand side show the pressures under a foundation supported by piles. The foundation chosen for the example was a square, 12 ft on a side. For the pile foundation, 16 piles were used. The piles were symmetrically placed, spaced at 3 ft on centers with 1.5 ft allowed between the pile centers of the outside rows and the edges of the foundation. A load of 10 tons per pile and a pile penetration of 40 ft below the bottom of the foundation were assumed. The piles were assumed to have a 14-in. butt and an 8-in. tip. In the formulas for pressure, a value of $R_c = 1.5$ was used. The spread foundation was the same size as the pile foundation and carried the same total load, 160 tons.

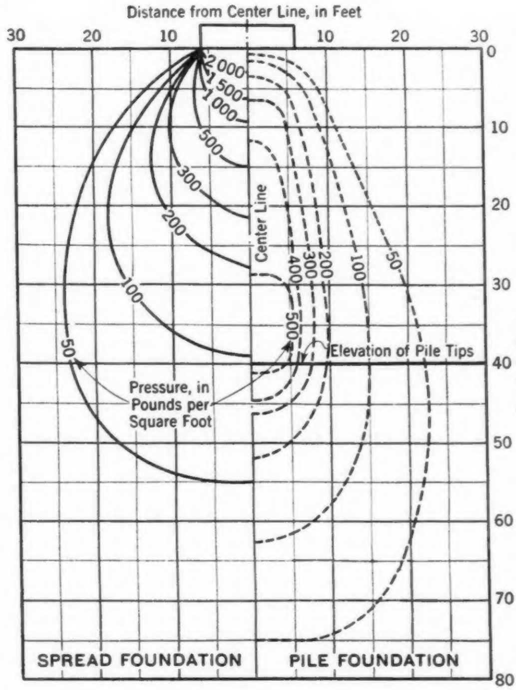


FIG. 12.—VERTICAL PRESSURE UNDER PILE AND SPREAD FOUNDATIONS

The pressures in the case of the pile foundation were computed from Eqs. 3c and 4b and were the summations of the pressures contributed by all of the piles. The pressures under the spread foundation were computed by the Boussinesq formula applied to subdivisions of the foundation area. The pressure contours are those in a vertical plane passing through the center of the foundation and perpendicular to the sides. It should be noted that this plane does not pass through any pile and does not show the local pressure concentrations close to the peripheries of the piles.

It is interesting to note the differences in pressure distribution. Under the assumed loading, the greatest pressure under the pile foundation does not

exceed 600 lb per sq ft and is confined to a zone around the lower 10 ft of the pile lengths, whereas, under the spread foundation, the pressures reach the unit load on the slab (2,220 lb per sq ft) with the higher pressures near the ground surface. In the usual case of a fairly uniform soil, it is of advantage to transfer the loads to considerable depths below the surface as soil samples generally will show a decreasing voids ratio with increasing depth. There are other cases, of course, in which the surface stratum is better than the underlying material. Under such a condition, a pile foundation might show more total settlement than a spread foundation of the same size.

The consolidation of soil held between piles that are spaced reasonably close together is undoubtedly a complicated procedure and probably not susceptible of analysis. It is possible that the entire block settles as a unit and that the pressure at the pile tips is the pressure of the foundation slab, less the pressure transferred to the surrounding soil by the outside rows of piles. Since, in general, the soil consolidation is proportional to the logarithm of the applied pressure, it would seem that piles in the ordinary soil formation would decrease the total settlement of a foundation considerably.

Corrections for *Transactions:* In *Proceedings* for November, 1941, transpose Figs. 7 and 10, leaving the captions as they are; and on page 1676, line 16, change the fraction to read "$\frac{P}{60 \times 0.92\,\pi}$." See also corrections, February, 1942, *Proceedings*, page 298.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

ENERGY LOSS AT THE BASE OF A FREE OVERFALL

Discussion

BY MESSRS. BORIS A. BAKHMETEFF AND N. V. FEODOROFF, CARL E. KINDSVATER, AND J. E. CHRISTIANSEN

BORIS A. BAKHMETEFF,[10] M. AM. SOC. C. E., AND N. V. FEODOROFF,[11] ESQ.[11a]—Mr. Moore's paper is a distinct contribution to engineering science. The subject is of great practical interest and the manner of presentation exemplifies well what modern "hydraulics" should be. Basic mechanical principles are comprehensively applied. Observed facts are reduced to generalized dimensionless curves. Above all, emphasis is laid on the physical aspects of the phenomena, recognizing the often baffling complexity of the ways in which water actually behaves, as compared with the simplified, antiquated notions.

In the course of its researches, the Fluid Mechanics Laboratory of Columbia University, in New York, N. Y., has gathered data bearing on the different topics presented in the paper. The experiments in part complement Mr. Moore's investigations in that they also embrace nonaerated nappes. Also, they help in further elucidating some of the physical features revealed in the paper, which at first sight may appear disconcerting.

An example in particular is the shape of the velocity profiles in the streaming below the fall as given in Fig. 6. The velocities shown are smallest at the surface and increase with the depth. There is no dynamical explanation for such "inverted" profiles in what otherwise appears to be more or less "parallel" streaming, and the source of the anomaly must be due to some undisclosed physical circumstance. In fact, the writers believe that the abnormalities are apparent, and are due to the fact that pitot-tube observations were made in the presence of air entrained by the falling nappe and gradually liberated in the subsequent streaming. In the course of escaping, air naturally concentrates in the upper layers, the density of which becomes substantially less than

NOTE.—This paper by Walter L. Moore, Jun. Am. Soc. C. E., was published in November, 1941, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: January, 1942, by Merit P. White, Assoc. M. Am. Soc. C. E.

[10] Prof., Civ. Eng., Columbia Univ., New York, N. Y.

[11] Research Associate, Columbia Univ., New York, N. Y.

[11a] Received by the Secretary February 13, 1942.

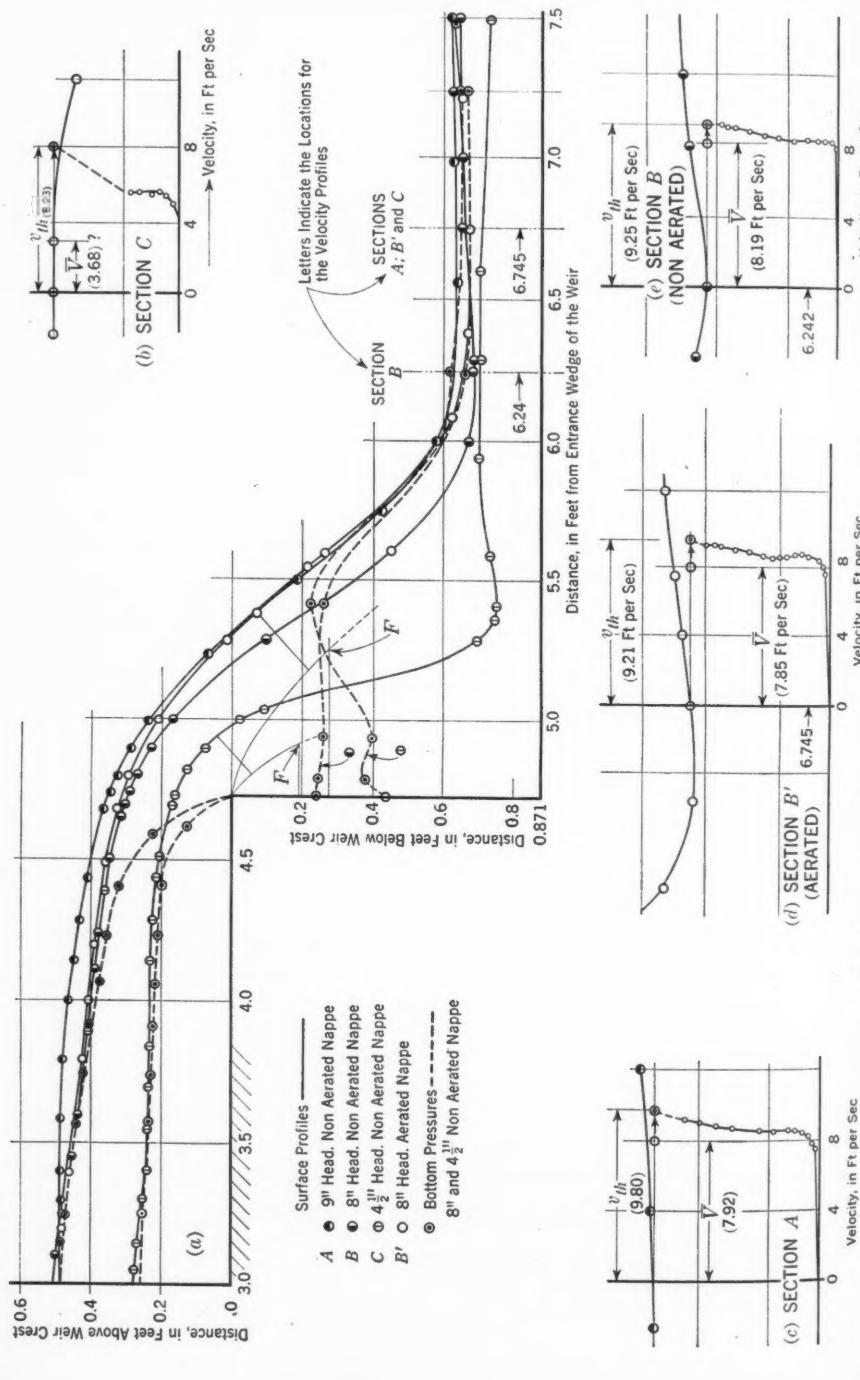


FIG. 20.—VELOCITY DISTRIBUTIONS OBSERVED BELOW A FREE OVERFALL

that of water. The manometer pitot readings, and the velocity calculations based on density of water, become misleading. Another source of error when operating with mixtures of air and water is the inevitable penetration of air into the manometer ducts.

These views are substantiated by Fig. 20, which typifies certain velocity distributions as observed below a free overfall. It also shows the corresponding surface profiles, the bottom pressure curves, and other particulars.

(The experiments devolve from researches in the hydraulics of broad-crested weirs. The overfall was formed by the lower extremity of a broad-crested weir, which had a length $L = 4.8$ ft and was erected in a 4-in. channel. Normally, the height of the weir body over the bottom of the channel was $h = 0.871$ ft. By inserting special floor panels into the lower channel, the vertical drop could be reduced to 0.136 ft—see legend in Fig. 21. The heads H, referred to

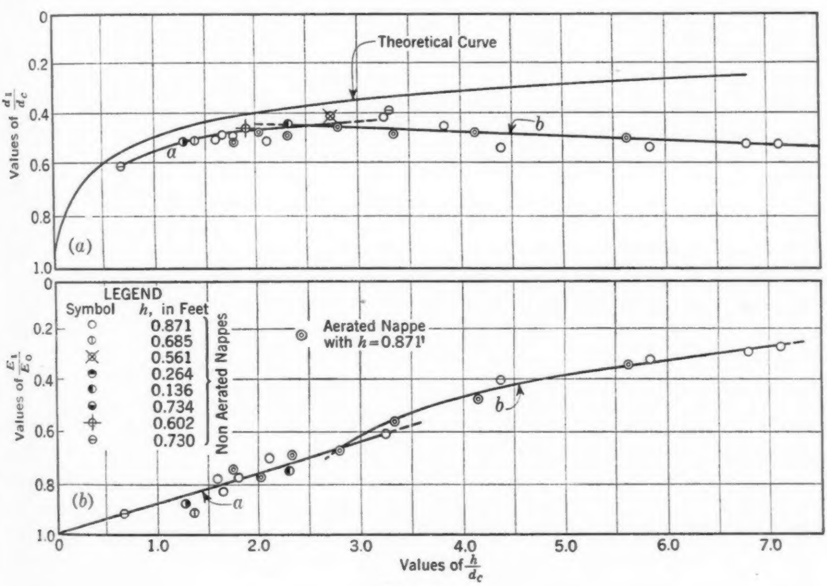


FIG. 21

in the figures, measure the elevation of the energy line in the approaching channel over the crest of the weir. The eventual discharges, conditioned by the $\frac{L}{H}$-ratio and the particular shape of the entrance wedge, were determined by means of a carefully calibrated Cipolletti weir and are featured by the parametric value of the critical depth. Special ducts communicating with the space under the nappe permitted the investigator to reproduce fully aerated as well as nonaerated conditions.)

(The velocity profiles were obtained by a pitot instrument having an impact tube diameter of $\frac{1}{16}$ in. The comparatively large size of opening permitted

the air bubbles to pass readily into the manometer tubes, and the disturbing influence of air was easily detected. In the experience of the laboratory, undersized impact tubes may become dangerously "sensitive" to air trouble.)

The velocity curve at section A (Fig. 20(c)) refers specifically to an unventilated nappe, with the vein sufficiently thick $\left(\frac{h}{d_c} = 1.77\right)$ for the "under-nappe" space to be completely filled with water. Entrainment of air from below is thus eliminated. Also, the streaming in the region next to the toe is comparatively undisturbed. As seen, the measured velocities offer no anomalies and, with the exception of a clearly defined friction zone near the bottom (the boundary layer), are rather uniform. In fact, near the surface the observed local velocities tend toward the "theoretical value"

$$v_{th} = \phi \sqrt{2\,g\,(h + 1.5\,d_c - d_1)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (26)$$

computed with $\phi = 1$. The partial decrease of the local velocities with the depth is attributable to the slightly concave outline of the streaming, which causes the internal pressures to exceed the hydrostatic. Under the particular circumstances, one may easily ascertain the average flow depth d_1 (which, as Mr. Moore rightly states, can scarcely be appraised when the motion becomes more tumultuous) and thus compute the average velocity $\bar{V} = \frac{q}{d_1}$, which proves to be about 0.85 v_{th}. Narrow channel forms, which accentuate losses, and the concavity of the filaments, account for such facts.

Altogether different circumstances, probably exceeding in intensity those reflected in Fig. 6, are exemplified by section C, Fig. 20(b). The vein in this instance is relatively thin $\left(\frac{h}{d_c} = 3.7\right)$, and the flow pattern is completely altered, as the unventilated nappe is changed to the "depressed type," with a block of air, in a state of underpressure, enclosed under the nappe. The surface line in Fig. 20(a) clearly exhibits how the excess atmospheric pressure pushes the falling vein toward the face of the drop, materially increasing the angle at which the falling water strikes the bottom. In fact, the streaming rebounds and splashes all over, offering on the whole a picture of intensely irregular and tumultuous conditions, well illustrated by Fig. 22. No reliable measurements can be made in the foaming mass near the toe, where the surface line in Fig. 20(a) conventionally indicates the lowest possible stage. Indeed, even at a subsequent station (section C, Fig. 20(b)), where the flow appears to be much calmer, the pitot instrument could be used with apparent success only in the lower regions of the profile. In the upper parts the continuous influx of air into the ducts made any reliable readings impossible. Note that the average velocity computed with the observed d_1, on the basis of the discharge being wholly liquid, is far less than the measured local v-values. On the other hand, the theoretical v_{th} exceeds the latter by far. Both circumstances are readily explained by the inflation of the streaming by the entrained air.

One would naturally assign air entraining action to the locus (curve F, Fig. 20(a)), where the lower surface of the falling vein "plunges" into the "under-nappe" roller. In fact, Fig. 1 very distinctly portrays the "mist" produced by the drawn-in air particles. If this were the only source of insufflation, nappes with blocks of air under them would behave very differently under all circumstances from their behavior when the under-nappe space is wholly filled with water. Observation shows that such is not the case, at least not entirely. Indeed, Figs. 20(d) and 20(e) exemplify velocity distributions obtained with the same discharge $\left(\frac{h}{d_c} = 2.01\right)$, but in one instance with an aerated under-nappe space (section B', Fig. 20(d)), and in the other (section B, Fig. 20(e)) with the space entirely filled with water.

The writers are led to believe that, for high $\frac{h}{d_c}$-values, air entrainment is caused additionally by the splashing and the generally tumultuous conditions that the motion assumes when a thin vein strikes the bottom at a large angle (Fig. 22).

FIG. 22

Whatever the cause and effect, one is led to accept the evidence that, in the instance of flow at the base of an overfall, just as in so many other structures, hydraulic analysis may become seriously handicapped by the presence of a new physical factor in the form of insufflated air, a factor which unfortunately is not easy to appraise and which totally upsets computations based on a purely liquid basis. Among other features, air entrainment naturally upsets energy appraisals. Under the heading "Previous Study," Mr. Moore refers to a recommendation by the senior writer[3] to ignore eventual losses in the design of stilling basins below falls and to use $\phi = 1$ in Eq. 2. Naturally, the suggestion at the time was in the nature of a "marginal" rule, prompted by a complete absence of experimental data. Also, as evidenced by sections A and B (Fig. 20), for relatively low falls the local surface velocities are actually not far from the theoretical frictionless values.

[3] "Hydraulics of Open Channels," by Boris A. Bakhmeteff, McGraw-Hill Book Co., Inc., 1932, p. 295.

For the phenomenon as a whole, however, and particularly for high $\frac{h}{d_c}$-values, the losses are very considerable, and probably should be ascribed to the intensive local turbulence engendered in the separation surfaces between the rebounding live vein and the underlying and the overlying rollers. (Most interesting flow patterns illustrating this case have been presented[12] by B. T. Morris, Jun. Am. Soc. C. E., and D. C. Johnson, Assoc. M. Am. Soc. C. E.) Under all circumstances it is interesting and useful to compare theoretical relations with those actually observed, even if the observations dealing with insufflated streaming must be accepted as necessarily approximate. As an addition to the valuable facts presented in Figs. 7 and 11, some of the Columbia observations are summarized dimensionlessly in Fig. 21. In Fig. 21(a), the ratio $\frac{d_1}{d_c}$, as observed, is compared with the theoretical values computed from

$$\frac{h}{d_c} = \frac{d_1}{d_c} + \frac{1}{2}\left(\frac{d_c}{d_1}\right)^2 - \frac{3}{2} \qquad (27)$$

which is equivalent to Eq. 4. In Fig. 21(b), the observed lower-channel specific energy

$$E_1 = d_1 + \frac{q^2}{2\,g\,d^2{}_1} \qquad (28a)$$

is compared with

$$E_0 = h + 1.5\,d_c \qquad (28b)$$

As Mr. Moore justly states, numerically the energy values (Fig. 21(b)) in the highly kinetic "shooting" state are extremely sensitive toward the d_1-values, and the latter by no means can be reliably appraised in the tumultuous current, crisscrossed by supersonic waves, etc., which obtains when $\frac{h}{d_c}$ becomes high; but, as stated, under such conditions the presence of entrained air inevitably baffles any attempts toward precision, and appraisals based on measuring the average depth may suffice for the purpose.

In the case of nonaerated nappes, the curves distinctly show two regimens. Curves a refer to relatively thick nappes $\left(\frac{h}{d_c} < 2.5\right)$, corresponding to flow forms where the nappe is wholly filled with water. Other experiments have demonstrated that the change from a fully filled under-nappe to the "depressed" form with an air block beneath it takes place in the region $2.2 < \frac{h}{d_c} < 2.7$, with an intermediary "unstable" reach between the indicated limits, where the one or eventually the other form may take place, depending on whether the flow is being raised or lowered.

Accordingly, the curves in the upper region (curves b) correspond to nonventilated flow with depressed nappes, with insufflation enhanced by the presence of air under the vein and by an increased angle of impact at the toe. As indicated in the legend (Fig. 21(b)), the measurements were made with largely

[12] *Proceedings*, Am. Soc. C. E., January, 1942, Figs. 5 and 6, pp. 26 and 27.

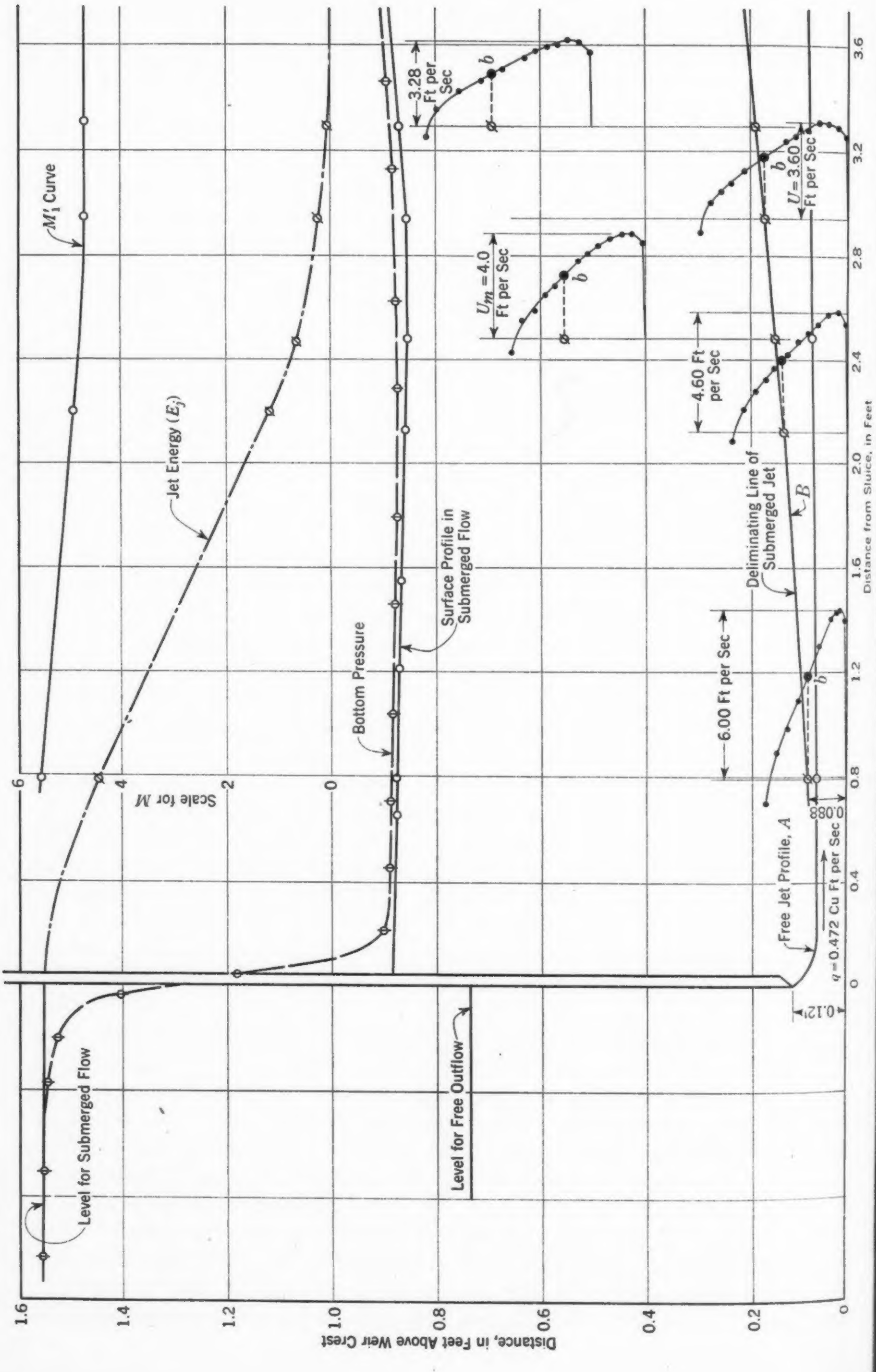
varying weir heights, so that the plotted points, particularly in the "reliable" low-drop zone, reflect a wide range of circumstances. The plottings also include observations relating to ventilated nappes with $h = 0.871$. Surprisingly, the points follow the same course closely.

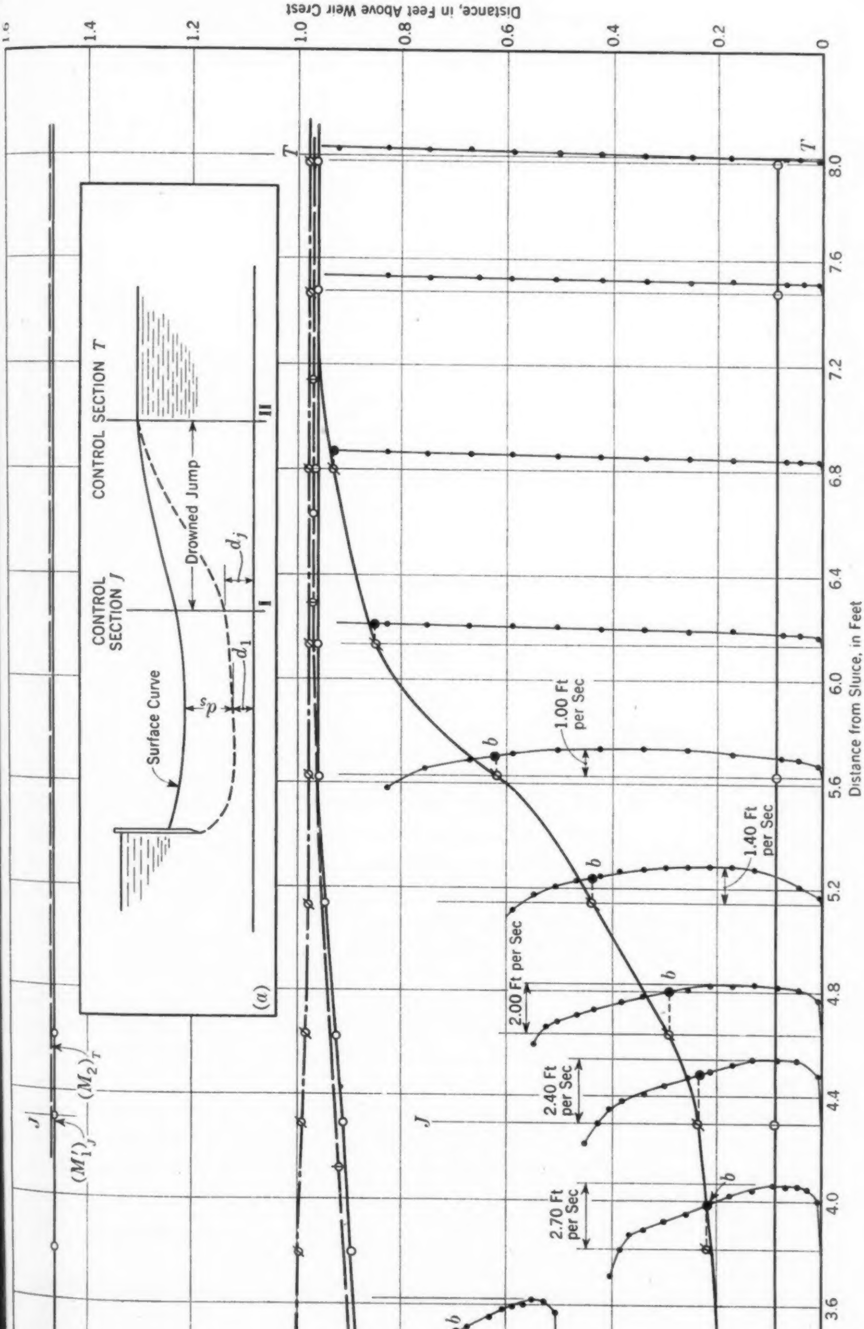
Obviously, the presence of air tends to exaggerate the distortions between the observed and the theoretical curves in Figs. 7 and 21. In fact, the measured velocities used for determining E_1 (see "Flow Conditions with Shooting Tailwater: Energy Loss at Base of Fall") are less than the actual values, just as the observed values of d_1, used in curve c and in Fig. 21(a), are artificially inflated. Thus, in either case the apparent energy losses are exaggerated.

Perhaps, under such circumstances, the most reliable and practical procedure would be to use curves of the type of Nos. (3) and (4), Fig. 11. This is particularly true if one were to associate the d_2-values with the eventual depth of the tailwater, which will balance the jump unfolding in proximity with the toe. Observations of such tailwater are reliable, in that they refer to a comparatively calm region, beyond the uncertainties of the insufflated zones. With regard to energy, an analogous rôle would be played by observed ratios of $\frac{E_2}{E_0}$.

A few closing remarks are due with regard to the very interesting observations relating to the effect of submergence, referred to under "Flow Conditions with Tranquil Tailwater: Effect of Submergence" and in conclusion (6) of the paper. Mr. Moore obviously has been dealing with a curious phenomenon, observed also on different occasions at Columbia University, and which in the parlance of the laboratory has been named the "drowned jump." The author's description may be complemented by Fig. 23, which relates to patterns observed in the flow from under a sluice. (The experiments, performed in a 6-in. flume, were conducted by Fred Ebbetsch, who at the time (1937–1938) was research assistant at the Columbia Laboratory.) As seen, when the heretofore free outflow (curve A, Fig. 23) becomes submerged by tailwater that exceeds the upper stage of a jump, the drowned jet continues for quite a while under the roller without much apparent change, slowly expanding until it reaches a certain section, section J, Fig. 23, beyond which it begins to rise rapidly, reaching the surface of the tailwater T on a steep gradient. In the part between the vena contracta and section J, the divergence is sufficiently low for the streaming to be treated as "parallel" in terms of the habitual Bernoulli equation, with the decreasing velocity adjusting itself to the increasing overpressure of the roller (d_s) and to the eventual energy losses. (The demarcation line B between the "jet" and the covering "roller" was traced by determining in the successive velocity profiles the points b . . . b, which delimit the discharge as it flows through the sluice. Computations were based on midstream conditions, as shown in Fig. 23, and obviously are approximate.)

Between sections J and T, Fig. 23, the rapid expansion resembles the outlines of a jump. In fact, the adverse slope of the vein between sections J and T is not far from the customary surface gradient of an "open jump" surface. Obviously, circumstances in the zone suggest the use of the "momentum principle." Lower or higher tailwater will shift the toe of the submerged jump


Level for Submerged Flow
Level for Free Outflow
M_1' Curve
Jet Energy (E_j)
Scale for M
Bottom Pressure
Surface Profile in Submerged Flow
Free Jet Profile, A
q = 0.472 Cu Ft per Sec
0.12
0.088
6.00 Ft per Sec
Deliminating Line of Submerged Jet
B
4.60 Ft per Sec
U_m = 4.0 Ft per Sec
U = 3.60 Ft per Sec
3.28 Ft per Sec
b
Distance from Sluice, in Feet
Distance, in Feet Above Weir Crest
0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
2.0
2.4
2.8
3.2
3.6
6
4
2



Distance, in Feet Above Weir Crest
Distance from Sluice, in Feet
CONTROL SECTION T
CONTROL SECTION J
Surface Curve
Drowned Jump
d_s
d_1
d_j
I
II
(a)
1.00 Ft per Sec
1.40 Ft per Sec
2.00 Ft per Sec
2.40 Ft per Sec
2.70 Ft per Sec
$(M_1')_J$
$(M_2)_T$
J
T
b

Fig. 23

(section J) away from the aperture, and vice versa, without, however, substantially modifying the general aspect of the situation.

When the phenomenon was first disclosed, it was natural to believe that the downstream extension of the jet would tend to endanger the floor of a structure over a considerable length, and that possibly an open jump would be a safer device from the point of view of energy dissipation. The argument, brought forward also by Mr. Moore, still stands in general and must await more detailed investigation. However, the poignancy is somewhat attenuated by the calculated outline of the jet energy featured in Fig. 23 by

$$E_j = (d_1 + d_s) + \frac{q^2}{2\,g\,d^2_1} \qquad (29)$$

As seen, the energy is rapidly lost in the course of expansion, being converted into turbulence in the process of intensive mixing. This turbulence develops between the live vein and the superimposed roller and is evidenced by the shape of the velocity profiles, indicating with what surprising rapidity the roller "eats" into the originally potential outflow.

To estimate, roughly, the possible location of the submerged jump, one may resort to the momentum principle by plotting the curve M'_1, featuring the quantity $\frac{q^2}{g\,d_1} + 0.5\,(d_1 + d_s)^2$ for the expanding jet (for designations see Fig. 23(a)), and comparing the latter with the value

$$M_2 = \frac{q^2}{g\,d_2} + 0.5\,d^2_2 \qquad (30)$$

computed for the tailwater.[13] To the exclusion of friction effects in the submerged jet, the values of $(M'_1)_j$ and $(M_2)_T$ for the control sections J and T should coincide. In the foregoing greatly simplified appraisal, it has been assumed that the positive and negative momentum flux in the circulating motion in the roller compensate each other. Also, the pressure distribution throughout the cross sections preceding section J is taken to be hydrostatic. It is indeed surprising that, under such rough "short cuts," the values of the calculated $(M'_1)_j$ and $(M_2)_T$ conform so well.

Carl E. Kindsvater,[14] Jun. Am. Soc. C. E.[14a]—Designers of hydraulic structures should find this paper of practical value. The author's use of dimensionless ratios in presenting test data reflects a growing tendency toward this practice.

Coordinated with previous research by Hunter Rouse,[15] Assoc. M. Am. Soc. C. E., this paper adds considerably to existing knowledge of the free overfall. An interesting comparison with Professor Rouse's experiments has been omitted by the author. Data presented with Fig. 6 give a value of 0.716 for the ratio

[13] "Hydraulics of Open Channels," by Boris A. Bakhmeteff, McGraw-Hill Book Co., Inc., 1932, p. 234.

[14] Asst. Hydr. Engr., Flood Control Section, TVA, Knoxville, Tenn.

[14a] Received by the Secretary March 2, 1942.

[15] "Discharge Characteristics of the Free Overfall," by Hunter Rouse, *Civil Engineering*, April, 1936, p. 257.

$\frac{d_0}{d_c}$. Similarly, from Fig. 10, $\frac{d_0}{d_c} = 0.714$. The depth d_c, used by the author as a reference parameter, should be defined as the nominal critical depth in the region of parallel flow upstream from the overfall. True critical depth, corresponding to the point of minimum specific energy, occurs at the brink of the overfall and is equal to the author's depth d_0. Professor Rouse found an average value of $\frac{d_0}{d_c} = 0.715$, which happens to correspond to the coefficient of contraction for a weir of zero height. It would be of interest to see a plot of all the author's data on this ratio.

In the seventh line following Eq. 6*b*, the author states: "With a given discharge and fall height, more effective dissipation may be achieved by increasing the width of the overfall section, and thereby reducing the value of d_c." The effect of spillway width upon energy dissipation is an important consideration in the selection of gates for large overflow dams. Fig. 24 shows one method of

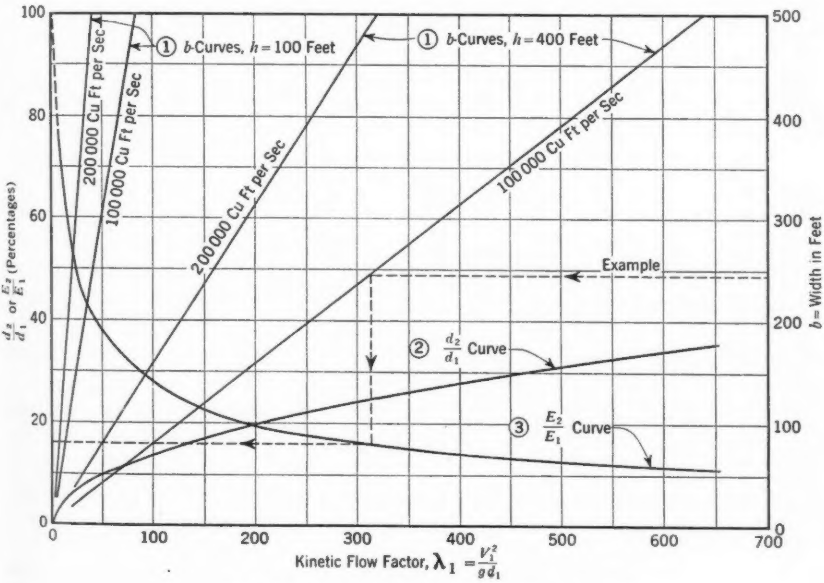


FIG. 24.—SPILLWAY PERFORMANCE CURVES

facilitating this choice. For illustration, consider an overflow spillway of width b, with a drop h between reservoir and tailwater levels. For preliminary study, it might be assumed that a perfect hydraulic jump will be caused to form at the foot of the spillway and that the lower conjugate depth, or the depth before the jump, will be reasonably close to $d_1 = \frac{Q}{b\,V_1} = \frac{Q}{b\,\sqrt{2\,g\,h}}$. The depth d_2 below the jump could be computed from Eq. 10*a* or 10*b*, but the

writer prefers to use an equation proposed by Messrs. Bakhmeteff and Matzke,[16]

$$\frac{d_2}{d_1} = \frac{1}{2}(-1 + \sqrt{1 + 8\lambda_1}) \quad \text{.................. (31)}$$

in which λ_1 is the "kinetic flow factor," analogous to Froude's number as a criterion for dynamic similitude. For the present case, the kinetic flow factor can be computed directly from the data given, or

$$\lambda_1 = \frac{V^2{}_1}{g\, d_1} = \frac{2\, b\, h^{3/2}\, \sqrt{2\, g}}{Q} \quad \text{.................. (32)}$$

Fig. 24 shows curves (1) in which λ_1 is plotted against b for values of h equal to 100 and 400 ft and values of Q equal to 100,000 and 200,000 cu ft per sec. Curve (2) in Fig. 24 is a curve of λ_1 against the ratio $\frac{d_2}{d_1}$ from Eq. 31. If, as defined by the author, E_1 and E_2 are the specific energies of flow before and after the jump, then $\frac{E_2}{E_1}$ represents the "efficiency" of the jump, a measure of energy dissipation below the dam. Curve (3) in Fig. 24 shows $\frac{E_2}{E_1}$ as a function of λ_1. Since curves (2) and (3) are applicable for all values of Q, b, and h, Fig. 24 demonstrates the practical advantages of the kinetic flow factor as a reference parameter.

The author's data showing a slight excess of pressure head over depth in the region of the toe of the jump are interesting. Close agreement or considerable disagreement between observers of such data may indicate only similarities or differences in laboratory technique. Tests on the hydraulic jump in a sloping channel might afford a better comparison with the author's data than tests on the level-floor jump. Messrs. Bakhmeteff and Matzke,[17] working with slopes as steep as 1 on 10, concluded, "* * * no appreciable difference can be observed between the depths and the local bottom pressures, except for a very limited zone near the toe of the jump * * *." Similar unpublished data taken by the late D. L. Yarnell, M. Am. Soc. C. E., for slopes from 1 on 6 to 1 on 1, generally show fair agreement between piezometer and point-gage profiles. Because of considerable admixture of air within the jump, it might be expected that the point-gage depths would exceed depths measured by piezometers.

J. E. Christiansen,[18] Assoc. M. Am. Soc. C. E.[18a]—In the opinion of the writer, the usefulness of this very interesting paper would be improved by plotting, or expressing, all of the dependent variables as functions of the one independent variable, $\frac{h}{d_c}$. For example, to determine the depth of standing water behind the fall, d_f, or the height of the jump, d_j, for any given or assumed

[16] "The Hydraulic Jump in Terms of Dynamic Similarity," by Boris A. Bakhmeteff and Arthur E. Matzke, *Transactions*, Am. Soc. C. E., Vol. 101 (1936), p. 636.

[17] "The Hydraulic Jump in Sloped Channels," by Boris A. Bakhmeteff and Arthur E. Matzke, *Transactions*, A.S.M.E., Vol. 60, February, 1938, p. 115.

[18] Irrig. and Drainage Engr., U. S. Regional Salinity Laboratory, U. S. Dept. of Agriculture, Riverside, Calif.

[18a] Received by the Secretary March 2, 1942.

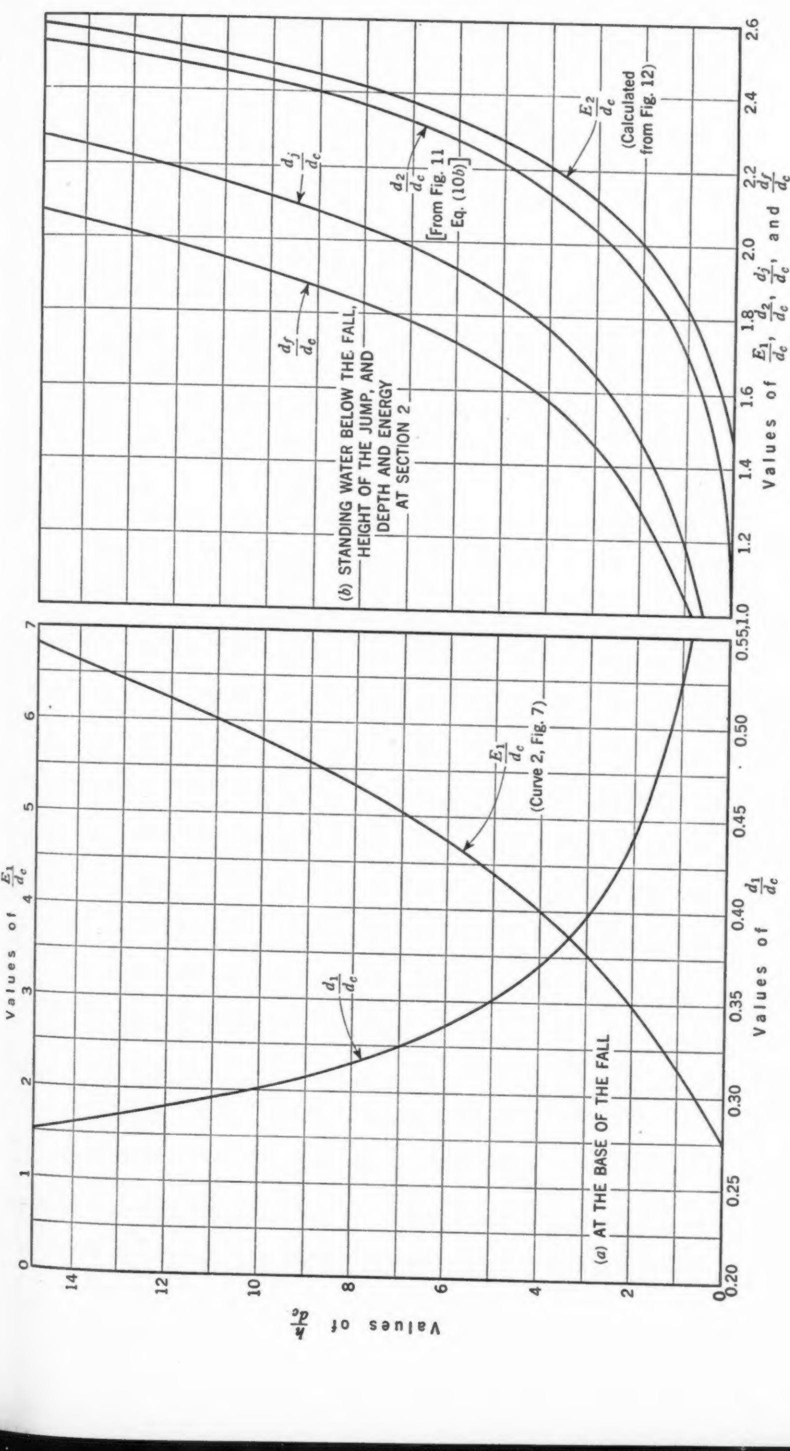


FIG. 25.—ENERGY AND DEPTHS BELOW THE FALL FOR VARIOUS RATIOS OF $\frac{h}{d_c}$

values of h and d_c, it is first necessary to determine d_1. To do this, one must select values of $\frac{E_1}{d_c}$ from curve (2), Fig. 7, and then solve a cubic equation (Eq. 9), which can be done graphically. Since $\frac{d_1}{d_c}$ is solely dependent upon $\frac{E_1}{d_c}$, which in turn is dependent upon $\frac{h}{d_c}$, why not plot $\frac{d_1}{d_c}$ against $\frac{h}{d_c}$? This has been done in Fig. 25(*a*), which also includes $\frac{E_1}{d_c}$, replotted from Fig. 7. The next step is to plot $\frac{d_f}{d_c}$ and $\frac{d_j}{d_c}$ directly against $\frac{h}{d_c}$. From Eqs. 8 and 9 and curve (2), Fig. 7(*a*), the relationship between $\frac{d_f}{d_c}$ and $\frac{h}{d_c}$ can be established. This has been plotted in Fig. 25(*b*). On this same figure, the values of $\frac{d_2}{d_c}$ have been replotted from Fig. 11. Values of $\frac{d_j}{d_c}$, being equal to $\frac{d_2}{d_c} - \frac{d_1}{d_c}$, are also shown. Likewise, for comparison, the relationship between $\frac{E_2}{d_c}$ and $\frac{h}{d_c}$ has been determined from Figs. 7(*a*) and 12, and is plotted in Fig. 25(*b*).

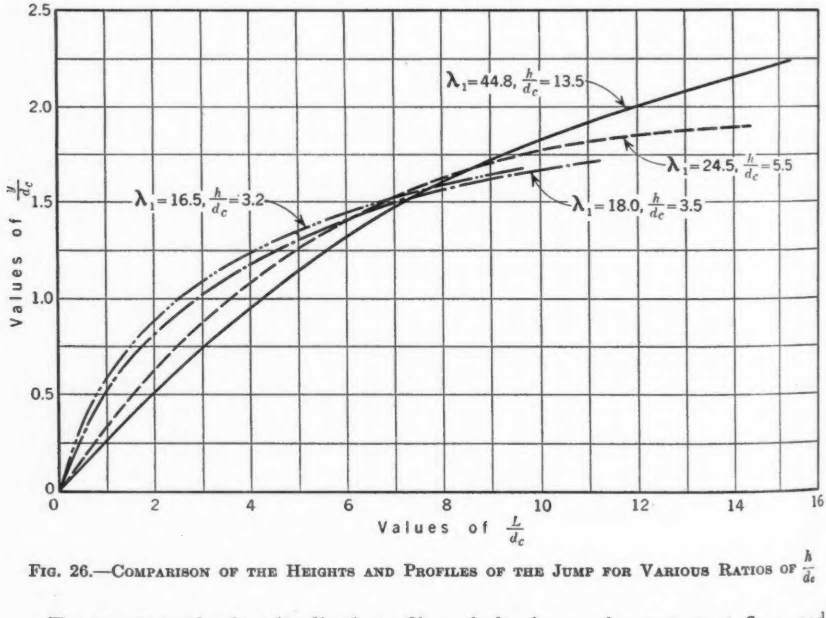


FIG. 26.—COMPARISON OF THE HEIGHTS AND PROFILES OF THE JUMP FOR VARIOUS RATIOS OF $\frac{h}{d_c}$

To compare the longitudinal profiles of the jump, for constant flow, and different values of h, the writer has plotted values of $\frac{y}{d_c}$ against $\frac{L}{d_c}$, as shown in Fig. 26. To do this it was necessary to calculate values of $\frac{d_1}{d_c}$, $\frac{h}{d_c}$, and $\frac{d_j}{d_c}$,

from the given values of λ_1, then select points on the curves in Fig. 13, and calculate points for the new set of curves. It would be interesting to know how the computed values of $\frac{h}{d_c}$ as given in Fig. 26 compare with the actual values.

Corrections for *Transactions:* In November, 1941, *Proceedings*, on page 1702, line 8, and page 1706, line 10, the summation should read "$\frac{\Sigma(V^3 \Delta A)}{2\,g\,\Sigma(V\,\Delta A)}$." Substitute the following corrected Fig. 12.

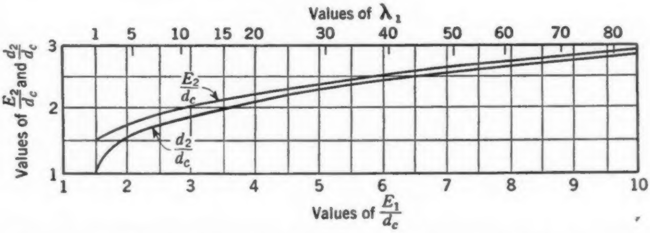


FIG. 12.—DEPTH AND ENERGY AFTER THE JUMP, PLOTTED AGAINST ENERGY BEFORE THE JUMP

In the Appendix, correct the notation as follows:

"L = length of jump measured from the beginning of the jump;
M = horizontal momentum; ΔM = change in horizontal momentum."

Furthermore, on page 1701, in the denominator of Eq. 5 change "d^2" to "d_1^2"; and substitute the following corrected Fig. 9.

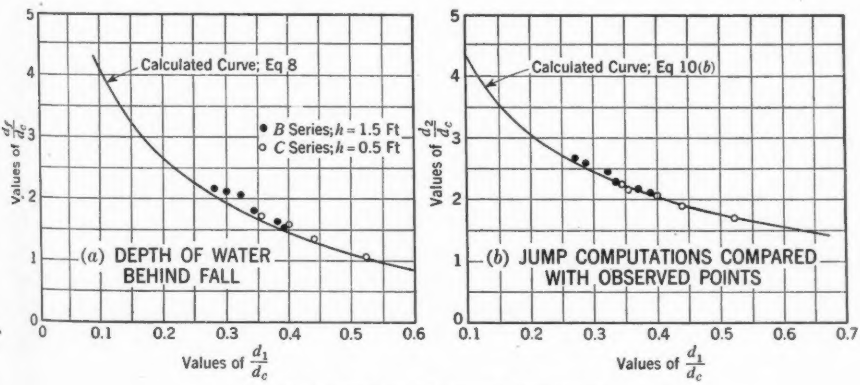


FIG. 9.—CURVES OF MOMENTUM EQUATIONS

values of h and d_c, it is first necessary to determine d_1. To do this, one must select values of $\frac{E_1}{d_c}$ from curve (2), Fig. 7, and then solve a cubic equation (Eq. 9), which can be done graphically. Since $\frac{d_1}{d_c}$ is solely dependent upon $\frac{E_1}{d_c}$, which in turn is dependent upon $\frac{h}{d_c}$, why not plot $\frac{d_1}{d_c}$ against $\frac{h}{d_c}$? This has been done in Fig. 25(a), which also includes $\frac{E_1}{d_c}$, replotted from Fig. 7. The next step is to plot $\frac{d_f}{d_c}$ and $\frac{d_j}{d_c}$ directly against $\frac{h}{d_c}$. From Eqs. 8 and 9 and curve (2), Fig. 7(a), the relationship between $\frac{d_f}{d_c}$ and $\frac{h}{d_c}$ can be established. This has been plotted in Fig. 25(b). On this same figure, the values of $\frac{d_2}{d_c}$ have been replotted from Fig. 11. Values of $\frac{d_j}{d_c}$, being equal to $\frac{d_2}{d_c} - \frac{d_1}{d_c}$, are also shown. Likewise, for comparison, the relationship between $\frac{E_2}{d_c}$ and $\frac{h}{d_c}$ has been determined from Figs. 7(a) and 12, and is plotted in Fig. 25(b).

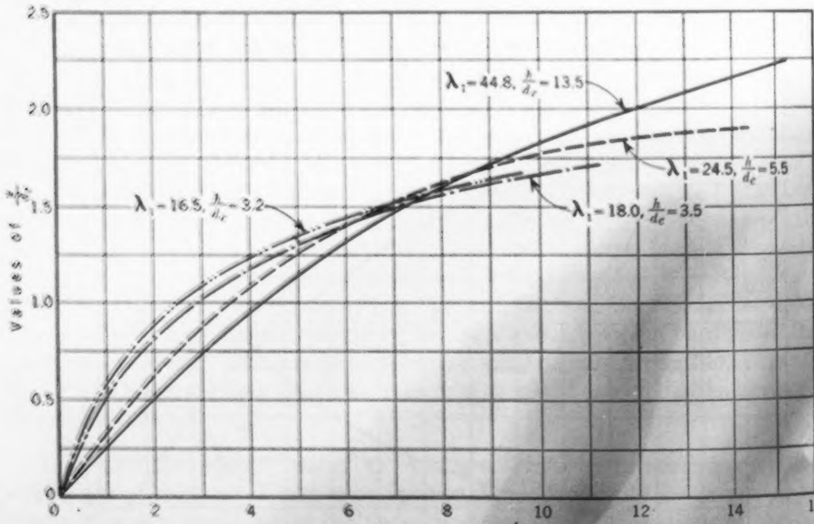


FIG. 26.—COMPARISON OF THE HEIGHTS AND PROFILES OF THE JUMP FOR VARIOUS RATIOS OF $\frac{h}{d_c}$

To compare the longitudinal profiles of the jump, for constant flow, and different values of h, the writer has plotted values of $\frac{y}{d_c}$ against $\frac{L}{d_c}$, as shown in Fig. 26. To do this it was necessary to calculate values of $\frac{d_1}{d_c}$, $\frac{h}{d_c}$, and $\frac{d_j}{d_c}$,

from the given values of λ_1, then select points on the curves in Fig. 13, and calculate points for the new set of curves. It would be interesting to know how the computed values of $\frac{h}{d_c}$ as given in Fig. 26 compare with the actual values.

Corrections for *Transactions:* In November, 1941, *Proceedings*, on page 1702, line 8, and page 1706, line 10, the summation should read "$\frac{\Sigma(V^3 \Delta A)}{2\,g\,\Sigma(V\,\Delta A)}$." Substitute the following corrected Fig. 12.

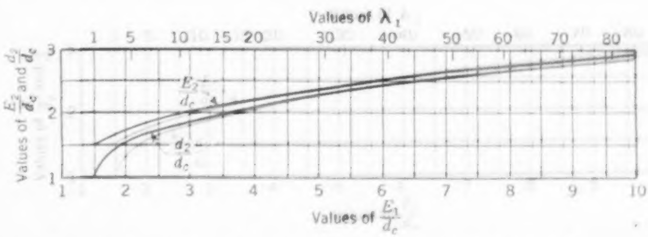


FIG. 12.—DEPTH AND ENERGY AFTER THE JUMP, PLOTTED AGAINST ENERGY BEFORE THE JUMP

In the Appendix, correct the notation as follows:

"L = length of jump measured from the beginning of the jump;
M = horizontal momentum; ΔM = change in horizontal momentum."

Furthermore, on page 1701, in the denominator of Eq. 5 change "d^2" to "d^2_1"; and substitute the following corrected Fig. 9.

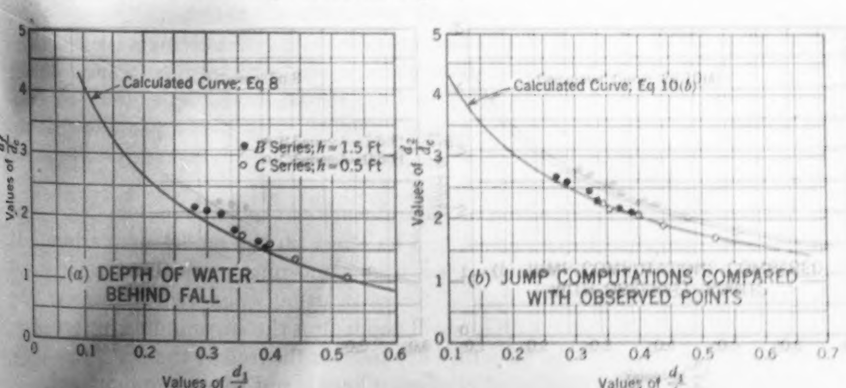


FIG. 9.—CURVES OF MOMENTUM EQUATIONS

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

HYDRAULIC DESIGN OF DROP STRUCTURES FOR GULLY CONTROL

Discussion

BY JOHN HEDBERG, ASSOC. M. AM. SOC. C. E.

JOHN HEDBERG,[11] ASSOC. M. AM. SOC. C. E.[11a]—The authors have done a very commendable job in establishing rules for the design of drop structures. It is especially gratifying to find that the rules have been founded on the basic fundamentals of flow rather than upon purely visual inspection.

However, there are several factors involved in the study that the authors have not chosen to discuss but which deserve some attention. The first of these is the use of a half section of the structure instead of the whole. The center wall interferes with the lateral movement of eddies, and therefore a better alinement of flows probably results in the model tested than in the corresponding prototype. Is it justifiable to dismiss this factor?

There is also a question of whether a model study involving a large amount of entrained air yields comparable results. The authors have varied dimensions to find the best combination in their model, based upon the flow characteristics of a combined air and water mixture. Is it not possible that the mixture will be quite different in the prototype and consequently have somewhat different flow characteristics?

There remains the question of the influence of the surrounding air pressure on the flow through the structure and its effect on the model comparison. It is so much trouble to run tests at less than atmospheric pressure that it is not surprising that strict similarity of forces is usually ignored. However, the shape and size of the ground roller will depend on the surrounding air pressure. Since the protection of the structure depends on this roller, there is a real question of whether the violation of similarity here is not of some importance.

Correction for *Transactions:* In Fig. 11(*a*), 11(*b*), 11(*c*), and 11(*d*), change "w_n" to "b_n."

NOTE.—This paper by B. T. Morris, Jun. Am. Soc. C. E., and D. C. Johnson, Assoc. M. Am. Soc. C. E., was published in January, 1942, *Proceedings*.

[11] Associate Prof., Civ. Eng., Stanford Univ., Stanford University, Calif.

[11a] Received by the Secretary March 4, 1942.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

STABILITY OF GRANULAR MATERIALS

Discussion

By Messrs. A. Hrennikoff, and D. P. Krynine

A. Hrennikoff,[9] Assoc. M. Am. Soc. C. E.[9a]—The novel approach to the problem of stability analysis of cohesionless granular materials, presented in this paper, is interesting.

The granular mass is visualized by the authors as divided into alternate elastic and plastic zones. The state of stress in the former obeys Hooke's law and is governed by the equations of equilibrium (Eqs. 12) and by the equation of compatibility or continuity, which for elastic materials takes the form of Eq. 3. The state of stress in the plastic regions is characterized by impending sliding at every point in the region, and is subject to the plasticity equation (Eq. 16) in addition to the equations of equilibrium (Eqs. 12), whereas Eq. 3 does not apply since the continuity has been broken. In these equations, allowance is made for the pressure of percolating water and for the constant horizontal earthquake acceleration. Expressions obtained for the stresses are linear functions of the coordinates.

The foregoing presents an outline of the proposed method, and now some of the limitations of the assumptions involved in it will be discussed.

1. Uniqueness of Solution.—A disconcerting feature of the problem of stress analysis in a mass of soil consisting of a combination of elastic and plastic regions is the multitude of the states of stress that may be found for the same exterior boundary conditions after some more or less artificial conditioning of the soil mass. This important feature distinguishes sharply the problem discussed here from the problems of the theory of elasticity, in which the solutions are unique.

To illustrate this point, consider a dam similar to the one shown in Fig. 4 and assume, for the sake of simplicity, that it is symmetrical about the vertical plane. Imagine, now, two adjacent thin, vertical, smooth walls cutting through the dam along the axis of symmetry and dividing it into two equal

Note.—This paper by R. E. Glover, Esq., and F. E. Cornwell, Esq., was published in November, 1941, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: November, 1941, by R. G. Hennes, Assoc. M. Am. Soc. C. E.; and March, 1942, by Stanley U. Benscoter, Jun. Am. Soc. C. E.

[9] Asst. Prof., Civ. Eng., Dept. of Civ. Eng., Univ. of British Columbia, Vancouver, B. C., Canada.

[9a] Received by the Secretary February 18, 1942.

parts. Holding the right wall for the time being in place, move the left wall horizontally to the left. A passive plastic state of stress forms now in the soil in the vicinity of the wall, and this plastic region extends gradually to the left, as the stresses increase with further advance of the wall. Next, leave the left side of the dam at the location it has reached and perform a similar operation on the right half by moving the right wall an equal amount. When the soil is so conditioned, eliminate the central gap between the two halves of the dam by moving one of the parts, together with its foundation, toward the other, and after that withdraw the walls. The two latter operations need not change the state of stress previously created in the two halves of the dam. Likewise wall movements may be imagined in the directions away from the soil creating an active plastic region in the middle of the dam. An infinite number of similar but unsymmetrical movements may also be conceived leading to a wide variety of the resultant states of stress, every one of which, no matter how artificial may be its origin, satisfies the same differential equations of equilibrium, compatibility, and plasticity, and the same exterior boundary conditions, as the solutions found by the proposed method.

Most of these theoretically conceivable states of stress bear no resemblance to the natural state resulting from the actual conditions of construction of the embankment, application of loads, and the character of possible failure.

It seems likely that the natural state of stress in the dam, as distinct from these artificial states, is also not unique in view of the influence of such rather indefinite factors as settlement, bulking, and shrinkage caused by temperature and moisture changes; but the range of such fluctuations in the natural state should not be wide.

This discussion leads to the conclusion that a solution of the differential equations cannot be accepted even as an approximate solution of the embankment problem, not to mention an exact solution, without at least demonstrating first by some appropriate qualitative analysis that such solution is reasonable under the given conditions. This qualitative investigation of the problem with the view to placing the elastic and plastic regions into locations where they are likely to develop, and of assigning to the latter the active or passive states as may be demanded by the circumstances, is completely overlooked in the method proposed. A stereotyped assumption of three regions, two plastic on the outside and one elastic between them, is used instead at every point of break in slope, and apparently under all circumstances—an assumption very convenient from the mathematical point of view, but totally unjustified physically.

Referring to the specific examples presented in Figs. 4 and 5, the presence of plastic states in regions I and VII above and below the dam seems particularly objectionable, resembling too much the generally discarded Rankine's theory of earth pressure. Also one does not see any reason for the existence of plastic states in zones III and V of Fig. 4. A dam with slopes as low as those shown in this example, and subjected to no loads other than its own weight, should, it would seem, be in an elastic state throughout.

If the stress pattern found by the proposed analysis may thus be deemed to "fall short of the mark," so the possible contention that the method answers

the question of stability or instability of the embankment must likewise be considered untenable in view of the arbitrariness of location of the elastic and plastic regions.

2. Linear Variation of Percolation Pressure.—Such linear variation is assumed throughout the granular mass, as is implied in the form of the expressions for the integration constants, Eqs. 13*a* and 13*b*, making the flow lines of the percolating water straight and parallel in each region. This simple but unrealistic assumption misrepresents the effect of water and greatly reduces danger to stability arising from convergence of the flow lines and from the related phenomena of boiling and piping at the toe of the dam.

3. Hooke's Law.—Some error results from the application of this assumption to the elastic regions for the following reasons:

(*a*) The stress-strain relationship for the soil is not quite linear;

(*b*) The curve of unloading does not completely coincide with the curve of loading;

(*c*) There is some uncertainty about the legitimacy of applying the compatibility equation to soil, since this condition implies preservation of elastic continuity of deformations throughout all the past history of the soil mass from the unloaded condition to its present state—that is, through the period of construction, settlement, and the following volume fluctuations caused by temperature and moisture changes.

This uncertainty is not removed by the fact that, in the method proposed, the compatibility condition does not contribute to the results, being satisfied automatically by any linear solution for stresses. It must be admitted, however, that compliance of the soil with Hooke's law has been postulated by many investigators.

4. Elastic and Plastic Regions.—When loading decreases or reverses, the regional boundaries between the elastic and plastic zones undergo some shifting. A region that was formerly plastic but is no longer so after the load change is neither elastic nor plastic in the sense used by the authors, since the limiting sliding condition exists no more, and at the same time the compatibility equation does not hold in view of the continuity having been broken during the plastic stage. This consideration seems to invalidate the applicability of the method to the conditions that involve a reversal or a substantial change in the character of the loading. Such conditions are present in an earthquake or in the case of a sudden drawdown in a reservoir retained by the earth dam—cases which the authors apparently list within the scope of their theory.

5. Continuity of Stresses.—The assumption of continuity across the elastic-plastic boundaries of the direct (normal) stresses parallel to the boundaries has not been substantiated and appears improbable.

Some minor inconsistencies and obscurities in presentation may also be mentioned.

In Example 2 it is difficult to interpret physical conditions corresponding to the solution given in Fig. 5. Judging by the pressure contours there seems to be a lack of equilibrium in the water conditions at the boundary of zones V and VI. Again, the part played by region V with regard to percolating water

is puzzling. This region seems to be composed of a very pervious material through which the water coming from region IV falls as through a chute. This naturally raises the questions: Where does this water go, and how can this situation be reconciled with the presence of pressure head in regions VI and VII?

Referring again to Fig. 5, it appears that the condition of equality of the boundary stresses at the border of zones II and IV is not satisfied. The same difficulty arises also in other places.

As the authors correctly point out, the state of stress in any elastic region must satisfy the inequality (see Eq. 16*a*) $R_0 > 0$. It is insufficient, however, to check this inequality at an arbitrary point, as the authors do, without proving that the function R_0 retains the same sign throughout the entire region.

It is not clear whether the data presented in Appendix II are based on the theory proposed; and, in case they are, how they are derived from it.

The foregoing reasons lead the writer to the belief that the proposed method of analysis of stresses in granular materials is unsatisfactory in view of its highly speculative nature and arbitrariness. However, this conclusion should not minimize the credit due the authors for the novelty of approach and mathematical ingenuity in attacking a baffling engineering problem.

D. P. Krynine,[10] M. Am. Soc. C. E.[10a]—The method of analysis described in this interesting paper consists in subdividing an earth dam into several regions by straight lines passing through the three vertexes of the triangle representing an idealized dam. These regions in alternate order are assumed to be "plastic" and "elastic," the difference between them being that the "plastic" regions are assumed to be on the point of losing "stability" by shear, whereas there is no shear danger in the "elastic" regions. As the authors themselves state, these terms do not define the type of the material. In fact, the latter, in general, is neither plastic nor elastic. It is regrettable that this inaccurate, though common, terminology has been adopted.

If an earth mass fails, it is generally due to a deficiency in lateral support. A vertical pressure requires a corresponding amount of horizontal pressure for stability. The "plastic" regions are brought to the state of incipient motion by so changing both the vertical and the horizontal pressures as to just satisfy Mohr's condition of failure.

Existing Methods of Analysis of Earth Structures.—The existing methods of analyzing earth structures may be subdivided into two main classes: (*A*) Methods by which stability is checked without computing stresses at all individual points of the earth mass; and (*B*) methods by which such stresses are computed.

Methods of Class (*A*).—A geological section of the locality should be traced, and the cross section of the earth structure located on it. For this purpose the general dimensions of the earth structure are first chosen mostly from experience.

From the study of the geological section and the cross section of the structure the designing engineer tries to answer the question: "What might happen

[10] Research Associate in Soil Mechanics, Dept. of Civ. Eng., Yale Univ., New Haven, Conn.

[10a] Received by the Secretary February 25, 1942.

to my structure?" As Hardy Cross, M. Am. Soc. C. E., frequently says in his lectures, this question should be answered by any designer analyzing his structure—in any branch of structural design. The most common case of possible failure of an earth structure is sliding along a more or less regular surface (failure line) which in cross section is generally assumed to be an arc of a circle (see Fig. 10). The analysis should be made for all possible cases of separation of part I of the structure (hereafter termed "wedge") from the remaining part II. Conditions of equilibrium for the case of incipient motion of wedge I are established, and the safety factor found.

This approach is deficient mostly because of unwarranted assumptions that must be made as to the distribution of the shearing stress and the shearing resistance along the assumed failure line AB. The methods of this class are very valuable, however, from several other points of view. Since wedge I is considered as a solid non-deformable body, no knowledge of stresses within it is needed. Furthermore, in establishing the shape of a failure line, the designer never loses sight of the actual physical conditions of both the structure and its foundation. An appreciable percentage of earth-structure failures are foundation failures, and one of the principal aims of the designer is to avoid them. Among the methods of class (*A*) the so-called "Swedish method" and the "ϕ-circle method"[11, 12] should be mentioned.

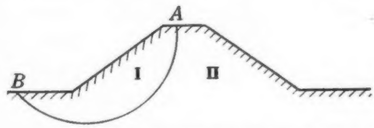


FIG. 10.—SEPARATION OF PART I OF A STRUCTURE FROM PART II ALONG THE FAILURE LINE AB

Methods of Class (B).—The writer limits himself to mentioning two methods of this class, both emanating from the Bureau of Reclamation. In one of them[13] the vertical normal stress, σ_y, equals the weight of the material above the given point, as in the Swedish method, and to take care of the horizontal normal stress a "compaction factor" is introduced. There are correction stresses designed to make the stresses satisfy the boundary conditions. Shearing stresses are checked against the shearing strength of the material, taking into account both friction and cohesion. In this way the factor of safety at different points of the dam is computed, and necessary corrections are introduced into the preliminary design of the structure.

The peculiarities of the method described in the paper as compared to the method by J. H. A. Brahtz, just outlined, are the following: (*a*) Cohesion is not taken into consideration; and (*b*) there is no evaluation of a factor of safety, but simply a determination of what the authors call "reserve of strength" in the "elastic" regions. Since the "plastic" regions used in this method are those in which a failure is likely to occur, the method in question approaches the methods of class (*A*) more closely than does that of Brahtz.

[11] "Contributions to Soil Mechanics 1925–1940," by Donald W. Taylor, published by Boston Soc. of Civ. Engrs., 1940, pp. 337–386; also, "Stability of Earth Slopes," by the same author, *Journal*, Boston Soc. of Civ. Engrs., Vol. XXIV, 1937, p. 197.

[12] "Analysis and Control of Landslides," by R. G. Hennes, *Bulletin*, Univ. of Washington, Eng. Experiment Station, 1936.

[13] "Rational Design of Earth Dams," by J. H. A. Brahtz, *Transactions*, 2d Cong. of Large Dams, Washington, D. C., 1936, Vol. IV, pp. 543–576.

Incipient Motion in "Plastic" Regions and Mohr's Circle.—If the stresses σ_x, σ_y, and $\tau_{xy} = \tau_{yx}$ at a point are in equilibrium, the ends of the vectors representing these values (σ_x = BC, σ_y = DE, $\tau_{xy} = \tau_{yx}$ = F′ E = CF, Fig. 11(*a*)) are located at a circle termed, as is well known, Mohr's circle. Vectors BC and

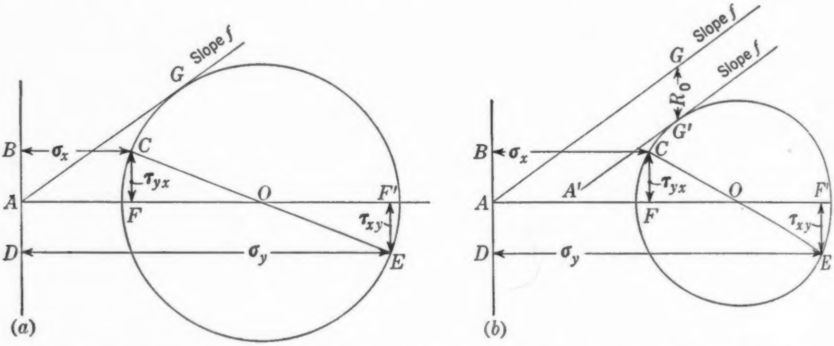


FIG. 11.—MOHR'S CIRCLE FOR: (*a*) A "PLASTIC" REGION; AND (*b*) AN "ELASTIC" REGION (GG′ = R_0)

DE are plotted horizontally from a vertical line BAD, A being a point located in the horizontal line passing through the center of the circle, O. Trace a line AG, its slope being equal to the coefficient of friction of the given earth material, *f*. In other words, make the angle GAO equal to the angle of friction. The point in question belongs to a "plastic" region if Mohr's circle touches line AG as in Fig. 11. If Mohr's circle is located below line AG and does not touch it, the point in question belongs to an "elastic" region.

Mohr's circle also may be constructed readily for the case of a cohesive material ($C > 0$). All formulas of the paper from Eq. 4*a* to Eq. 11*b*, inclusive, may be directly, or practically directly, seen on Mohr's circle. No doubt these equations are correct, but most of them as represented in an analytical form are not necessary for the purposes of the paper. In particular, R (Eq. 6*a*) is nothing else but the vertical distance between different points of Mohr's circle and line AG. Its minimum value, function R_0 (Eq. 6*b*), is the vertical distance between parallel lines AG and A′G′, the latter being tangent to the circle (Fig. 11(*b*)).

The values of the vertical pressure, σ_y, and the horizontal pressure, σ_x, to be used in the construction of Mohr's circle, should be corrected for percolation and earthquake. The vertical and horizontal components of the force of percolation, per unit of volume of the structure, are denoted by $\frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial x}$ in the paper. The vertical component $\frac{\partial u}{\partial y}$ ("uplift") acts against gravity. The symbol C_2 as used in the paper is the unit weight of dry earth material, plus the weight of moisture in pores, if any, minus uplift if any (Eq. 13*a*).

The horizontal component of the percolation force, $\frac{\partial u}{\partial x}$, and the horizontal force of earthquake, h, per unit of volume of the dam, decrease the lateral

support. The algebraic sum of these two quantities, designated by $C_1 C_2$ in the paper, is given per unit of volume of the dam. It must be recomputed per square unit of the cross section of the dam as shown hereafter.

Differential Equations Are Not Necessary in "Plastic" Regions.—The use of differential equations to compute stresses in "plastic" regions may be avoided by using the following simple procedure. Refer to Fig. 12(a) in which x and y correspond to the point where stresses are to be computed. Cut from the dam a vertical prism ABCD having DC = 1 sq unit for base, this "unit" being

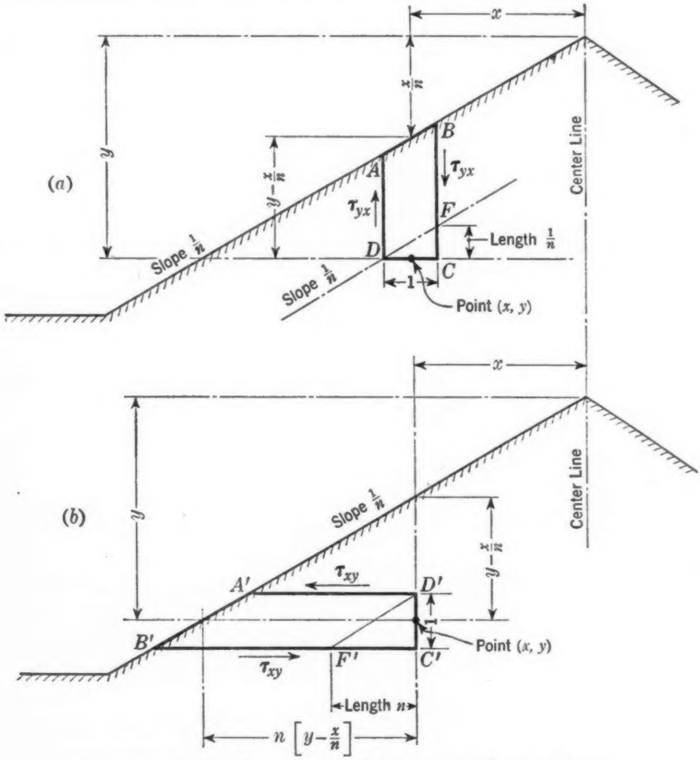


FIG. 12.—NORMAL STRESSES AND SHEARS IN A "PLASTIC" REGION

very small. The given point is supposed to be at the middle of this base. The latter carries the vertical volume forces of the prism ABCD, which amount to $C_2\left(y - \frac{x}{n}\right)$, this being the absolute value of those forces. In addition, there is an excess of shearing stress, τ_{yx}, acting along area FC $= \frac{1}{n}$ sq units, since the shearing stresses along areas AD and BF are balanced. Hence, the vertical pressure, σ_y, at point (x,y), is:

$$\sigma_y = -C_2\left(y - \frac{x}{n}\right) + \frac{1}{n}\tau_{yx} = \frac{C_2}{n}(x - n\,y) + \frac{\tau_{yx}}{n} \qquad (36a)$$

In a similar manner, Fig. 12(*b*) represents a horizontal prism A′B′C′D′ cut from the dam, the given point (x,y) being at the middle of its base D′C′ = 1 sq unit. Fig. 12(*b*) is a complete analogy of Fig. 12(*a*), and the horizontal pressure at point (x,y) is:

$$\sigma_x = C_1\, C_2\, n \left(y - \frac{x}{n} \right) + n\, \tau_{xy} = - C_1\, C_2\, (x - n\, y) + n\, \tau_{xy} \ldots (36b)$$

Because of the law of equality of action and reaction, it is sufficient to study only one half of the dam as in Fig. 12.

Remembering now that $\tau_{yx} = \tau_{xy}$, multiply Eq. 36*a* by n^2 and subtract Eq. 36*b* from Eq. 36*a*, thus modified, eliminating τ_{xy}. For given values of the coefficient of friction f and of the slope n, and if $C_1 = 0$, the pressures σ_y and σ_x are interrelated, thus:

$$\sigma_y = K\, \sigma_x \ldots (37)$$

in which K is a constant to be measured on Mohr's circle (Fig. 13). First draw a horizontal line AO and a line AG with a slope $f = 0.7$ (as in the ex-

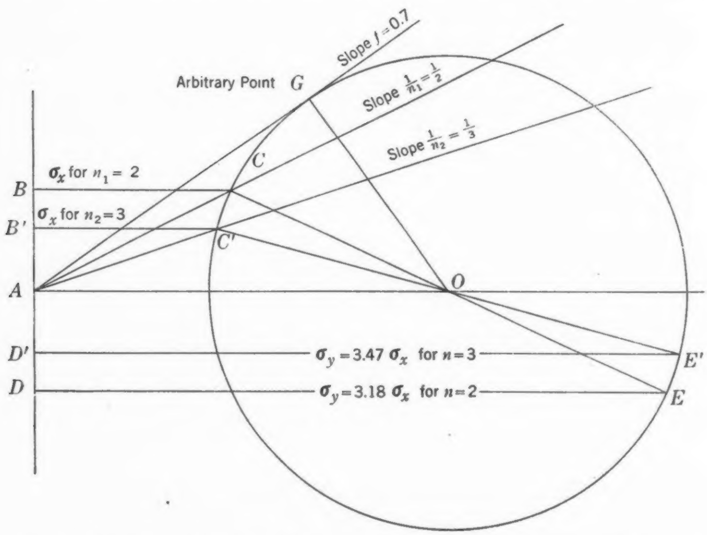


FIG. 13.—RATIO BETWEEN THE VERTICAL AND THE HORIZONTAL NORMAL STRESSES IN A "PLASTIC" REGION (CASE $C_1 = 0$; FIGS. 4 AND 5)

amples of the paper). A perpendicular line GO at an arbitrary point G of line AG determines the position of the center, O, of Mohr's circle. The two lines AC $\left(\text{slope } \frac{1}{n_1}\right)$ and AC′ $\left(\text{slope } \frac{1}{n_2}\right)$ determine vectors BC and B′C′ corresponding to the horizontal pressures σ_x. In their turn, vectors DE and D′E′ correspond to the vertical pressures, σ_y.

If C_1 is not equal to zero, the dam may be visualized as having a steeper slope $\frac{1}{n_0}$ and being at the "plastic" state under the action of the vertical body forces only. Let:

$$\sigma_x = n_0 \tau_{xy} \quad \text{(38)}$$

from which, using Eq. 36*b*:

$$\tau_{xy} = \frac{C_1 C_2}{n - n_0}(x - n y) \quad \text{(39a)}$$

and

$$\sigma_x = \frac{n_0 C_1 C_2}{n - n_0}(x - n y) \quad \text{(39b)}$$

Introducing the value of τ_{xy} from Eq. 39*a* into Eq. 36*a* and dividing Eq. 36*a*, thus modified, by Eq. 39*b*:

$$\frac{\sigma_y}{\sigma_x} = \frac{n - n_0 + C_1}{n\, n_0\, C_1} \quad \text{(40)}$$

Compute the values: $\frac{\mathrm{DE}}{\mathrm{BC}}$ and $\frac{n - n_0 + C_1}{n\, n_0\, C_1}$, in which n and C_1 are given, and $n_0 = \frac{\mathrm{BC}}{\mathrm{CF}}$, for various points C as in Fig. 11(*a*). Plot these values from points C outward, along the radii of Mohr's circle. The intersection of curves I and II thus obtained (Fig. 14) furnishes the true value of the ratio $\frac{\sigma_y}{\sigma_x}$. A simple trial-and-error procedure would also serve the same purpose.

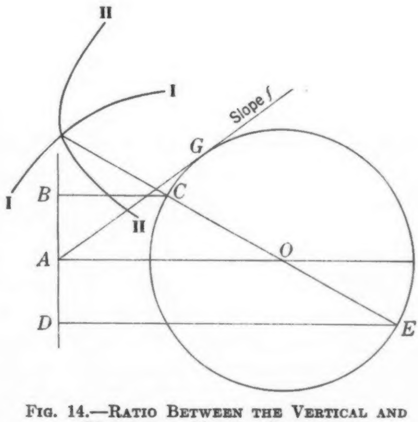


FIG. 14.—RATIO BETWEEN THE VERTICAL AND THE HORIZONTAL NORMAL STRESSES IN A "PLASTIC" REGION ($C_1 \gtreqless 0$)

It is easy to see that Eqs. 36 and 37 furnish the same results as Eqs. 21. In the case of a "plastic" region such as I or VII, in which $n = \infty$, the Rankine formula should be applied (Eqs. 22), and no special derivation is needed.

Stresses in "Elastic" Regions.—It is postulated in the paper that the material in the "elastic" regions obeys Hooke's law, which is not true. Afterward the condition of compatibility is used under the form of Eq. 3, which presumes that the material obeys Hooke's law. Using this condition, it is found that the stress distribution in "elastic" regions is rectilinear. This result is all that is needed for finding stresses in the "elastic" regions. As soon as the stresses at the boundaries of the plastic regions at a certain elevation are computed, the stresses in the "elastic" regions at the same elevation may be found simply by tracing straight lines as in Figs. 4 and 5, since equality (not "continuity" as the authors state) of stress across a boundary is postulated. Under the

circumstances, it would be much simpler to postulate, directly, the rectilinear stress distribution in the "elastic" regions.

Slip Lines.—Instead of using Eqs. 24 and 25, the slip lines in a boundaryless "plastic" region may be located simply from Mohr's circle (Fig. 11). The major principal stress, σ_1, makes half the angle F′OE with the vertical. The two systems of slip lines make the angle $45° - \frac{1}{2}$ arc tan f with the direction of the major principal stress, σ_1, on both sides of it.

Title of the Paper.—The writer believes that the title of the paper is inappropriate. It is very questionable whether this method can be applied in the case of embankments made of a material different from the foundation, especially on a transverse slippery slope. Such is the case of many highway and railway embankments, dikes, and levees.

Conclusion.—In the opinion of the writer, the method described in this paper is interesting and original. Unfortunately, the method in question, although simple in itself, has been developed and presented in such a manner that quite a screen of mathematical symbols obscures its physical significance.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

VARIATION OF ELASTIC CHARACTERISTICS ON STATICALLY INDETERMINATE QUANTITIES

Discussion

By A. H. Finlay, Assoc. M. Am. Soc. C. E.

A. H. Finlay,[4] Assoc. M. Am. Soc. C. E.[4a]—Professor Hrennikoff has presented an intriguing paper with clarity and precision. Others[5] previously have determined the effect of variations of the elastic characteristics of structures upon the statically indeterminate stress functions, but these workers have confined themselves to arbitrary distributions of such variations throughout the structure. To the writer's knowledge, the author is the first to develop a method determining the maximum effect that a variation of specified magnitude may exert upon a given stress function. In so doing he defines the "pattern" to which such a variation must conform.

The author's applications of his method to several specific structures reveal that substantial variations in the stress functions are at least within the realms of possibility and the profession is fortunate to have the problem placed upon a quantitative basis.

In order to interpret the significance of the expressions under the integral signs in the author's various equations (such as Eq. 30*a*), the writer has plotted the functions concerned for a special case—a square frame having members of uniform cross section and carrying a uniform load on the horizontal member. In Fig. 8, curves have been plotted for those expressions associated with dX_a, dX_b, and dM_E, defined, respectively, as follows:

$$dX_a\,(\max) = \frac{w\,L^2\,i}{378}\,A_a \qquad (38a)$$

$$dX_b\,(\max) = \frac{w\,L^2\,i}{756}\,A_b \qquad (38b)$$

Note.—This paper by A. Hrennikoff, Assoc. M. Am. Soc. C. E., was published in January, 1942, *Proceedings*.

[4] Associate Prof., Civ. Eng., Univ. of British Columbia, Vancouver, B. C., Canada.

[4a] Received by the Secretary March 2, 1942.

[5] "Dependability of the Theory of Concrete Arches," by Hardy Cross, *Bulletin No. 203*, Univ. of Illinois Eng. Experiment Station, Urbana, Ill., 1930.

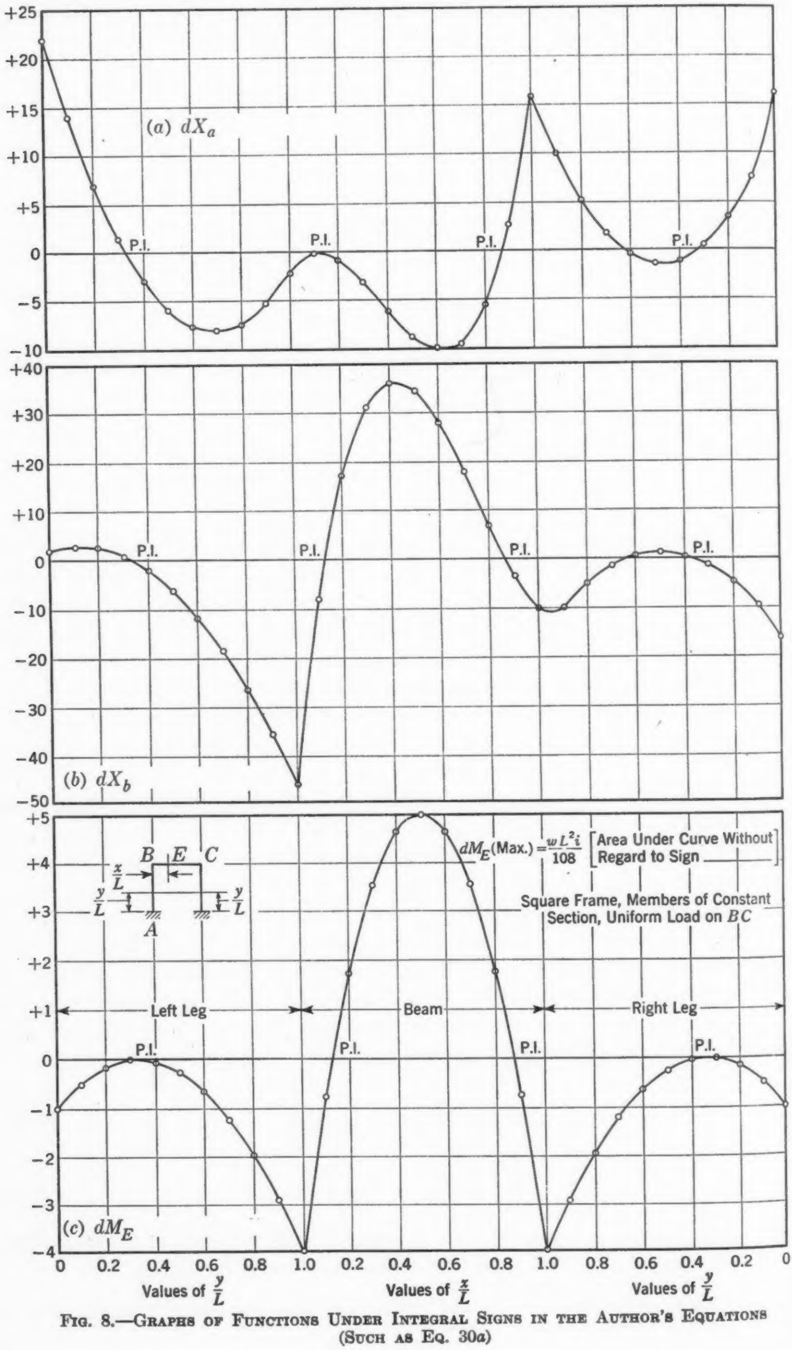


FIG. 8.—GRAPHS OF FUNCTIONS UNDER INTEGRAL SIGNS IN THE AUTHOR'S EQUATIONS (SUCH AS EQ. 30*a*)

and

$$dM_E\,(\text{max}) = \frac{w\,L^2\,i}{108}\,A_E \qquad (38c)$$

in which A is the area under each respective curve, without regard to sign. (The graph of the function under the integral for dX_c is similar to Fig. 8(b), but "opposite hand.") These areas are, of course, measures of the maximum values that dX_a, dX_b, and dM_E may attain. It should be noted that the areas under these curves with regard to sign are zero since a uniform change in an elastic characteristic (that is, with constant i) produces no change in the statically indeterminate forces. It is the fact that i may be positive in those parts of the structure where such areas are positive, and negative in those parts where the areas are negative (or vice versa), that leads to the possibility of the substantial errors found by the author. It is also apparent from the curves that considerable errors may still occur in the stress functions even though chance fails to cause a change in the sign of i in strict accordance with the pattern. For example, in the case of M_E, should i chance to be of either sign between the points of inflection of the horizontal member and zero at all other points, the error in M_E will amount to about half its maximum value.

The author's equations, of course, may be used to determine errors introduced by cracking. In this connection the curves yield some interesting information without the labor involved in evaluating the integrals. It will be noticed that the particular curves plotted all pass through zero at or near the points of inflection for the loading considered. This may not be true in general, however. Consider dX_b produced by cracking: If i, due to cracking, is taken as constant for all parts of the structure except in the vicinity of the points of inflection (in which latter sections, of course, it will be zero), it is seen that a measure of dX_b will be obtained by the area, this time taken with regard to sign, under all those parts of the curve, Fig. 8(b), where i is not zero. As zero values of i must be confined to relatively short lengths of the structure in the vicinity of the inflection points, and since the area under the entire curve with regard to sign is zero, it is clear from the shape of the curve that the remaining area, and hence the value of dX_b due to cracking, must be small. Similar reasoning in the case of dX_a and dM_E leads to the same conclusion. Values of these quantities have been found on the assumption that a uniform i results from cracking at all sections where the bending moment is greater than one quarter of the design value. This is quite arbitrary, of course, and has been used for illustrative purposes. The values are as follows:

$$dX_a = +\,0.0358\,i\,X_a \qquad (39a)$$

$$dX_b = -\,0.00573\,i\,|\,X_b\,| \qquad (39b)$$

and

$$dM_E = -\,0.00321\,i\,M_E \qquad (39c)$$

It should be noted that, in Eqs. 39, i carries a sign and, since cracking reduces the moment of inertia, i should be substituted in Eqs. 39 as a negative quantity. The errors are seen to be extremely small, as could have been anticipated from the curves.

Had i been assumed as a variable quantity, the curves in Fig. 8, of course, would have been different, since i could not have been left outside the integral sign. As an approximation, however, the curves as plotted could be used in such a case. The areas under various parts of them then would have to be multiplied by the value of i assumed to exist over the section in question. This, of course, is equivalent to assuming that i is constant along short sections of the structure but varies abruptly from section to section. It still seems clear, however, from the characteristics of the curves and the fact that i is everywhere of the same sign, that errors in the statically indeterminate forces due to cracking will be of a smaller order altogether than those which may arise due to chance variations in the elastic characteristics.

The writer is in complete agreement with the author's concluding paragraphs, and he also feels that the paper should prove of considerable interest to those whose lot it is to explain discrepancies between observed and calculated quantities in the field of rigid-frame testing.

Correction for *Transactions:* In January, 1942, *Proceedings*, page 71, change title of paper by A. Hrennikoff to read "Effect of Variation of Elastic Characteristics on Static Unknowns."